

# The New Real Number System: An Overview

## 1. 1 Introduction

Why do we need a new real number system? The real number system defined by the field axioms which is supposedly a complete ordered field has a number of defects. It is neither complete nor ordered nor a field but most of all the the field axioms are inconsistent, Felix Brouwer and this host have constructed counterexamples to the the trichotomy axiom; moreover, it suffers from the inherent ambiguity of infinite set and large and small number that make the completeness axiom false. It its not ordered by the relation “<” so necessary to put the real numbers on the real line for the purposes of analysis. All along we have assumed the linear ordering of the real numbers which is not true. There are a number of paradoxes that arise from the real number system the most famous of which is the Banach-Tarski paradox for which the axiom of choice, a variant of the completeness axiom, is blamed. (For a thorough account of the development the real number system and its foundations, the emergence of contradiction and paradoxes and its rectification see, The Hybrid Grand Unified Theory, by E. E. Escultura, V. Lakshmikantham, S. Leela, Atlantis, a division of Elsevier Science, Ltd.), 2009)

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## 1.2. Construction of the contradiction-free real number system

**The terminating decimals.** In the applications of the real numbers and in proving theorems we rarely use the field axioms but the decimals as base 10 place value numerals and when on rare occasions we use some of the field axioms, e.g., axiom of choice and the trichotomy axiom we run smack into a contradiction or paradox. Therefore, we reconstruct the real number system on the decimal numerals subject to three simple axioms and call it the new real number system  $\mathbf{R}^*$ :

**Axiom 1.**  $0, 1 \in \mathbf{R}^*$ .

**Axiom 2 & 3.** The addition and multiplication tables  $+$ ,  $\times$  of arithmetic. The addition and multiplication tables well-define the additive and multiplicative identities 0, 1, i.e., specify their properties or behavior, and construct the integers and terminating decimals using 0 and 1. Since we start with finite elements and operations no uncertainty exists in the initial construction.

Then we define the integers as base 10 place-value numerals:

$$a_n a_{n-1} \dots a_1 = a_n 10^n + a_{n-1} 10^{n-1} + \dots + a_1, \quad (1)$$

where the  $a_n$ s are the basic integers, 0, 1, ..., 9.

Now, we extend the integers to include the additive and multiplicative inverses  $-x$  and, if  $x$  is not 0,  $1/x$  (reciprocal of  $x$ ), respectively. Note that the reciprocal of an integer exists only if it has no prime factor other than 2 or 5. This extension requires corresponding extension of the operations  $+$  and  $\times$ , in effect, re-stating associativity, etc., as part of its axioms and something else that is new: the rules of sign that we take

as part of the axioms of this extension (we need not write them as they are familiar). Then we define a new operation: division of an integer  $x$  by a nonzero integer  $y$ , or quotient, denoted by  $x/y$  and defined by:

$$x/y = x(1/y). \quad (2)$$

This quotient exists only when  $y$  has no prime factor other than 2 or 5. We similarly extend associativity and commutativity of addition and multiplication and distributivity of multiplication relative to addition and include them as axioms of the extension. We consider subtraction the inverse operation of addition and division that of multiplication, examples of duality that we shall consider in detail below.

We define a terminating decimal as follows:

$$\begin{aligned} a_n a_{n-1} \dots a_1 \cdot b_k b_{k-1} \dots b_1 &= a_n 10^n + a_{n-1} 10^{n-1} + \dots \\ &+ a_1 + b_1/10 + b_2/10^2 + \dots + b_k/10^k \\ &= a_n 10^n + a_{n-1} 10^{n-1} + \dots + a_1 + b_1(0.1) \\ &+ b_2(0.1)^2 + \dots + b_k(0.1)^k. \end{aligned} \quad (3)$$

where  $a_n a_{n-1} \dots a_1$  is the integral part,  $b_1 b_2 \dots b_k$  the decimal part and  $0.1 = 1/10$ . Note that the terminating decimals are well-defined since the reciprocal of 10 has only the factors 2 and 5. Then the quotient  $x/y$  of an integer  $x$  by nonzero integer  $y$  exists only if  $y$  has no prime factor other than 2 or 5. Such quotient is called rational. We recall that in the real number system a rational is nonterminating periodic (a terminating decimal is periodic). This is ambiguous for nonterminating decimal since it is not verifiable. We call a decimal that is not rational *errational* but we do not know what that is yet because we know nothing beyond the terminating decimals.

This definition of the integers as the integral parts of the terminating decimals resolves the inadequacy of Peanos's postulates in the development of the natural numbers for they are clearly isomorphic to them and makes them integers.

**The nonterminating decimals.** Now we define the nonterminating decimals for the first time without contradiction and with contained ambiguity, i.e., approximable by certainty. We build them on what we know: the terminating decimals, our point of reference for all its extensions.

A sequence of terminating decimals of the form,

$$N.a_1, N.a_1 a_2, \dots, N.a_1 a_2 \dots a_n, \dots \quad (4)$$

where  $N$  is integer and the  $a_n$ s are basic integers, is called standard generating or  $g$ -sequence. Its  $n$ th  $g$ -term,  $N.a_1 a_2 \dots a_n$ , defines and approximates its  $g$ -limit, the nonterminating decimal,

$$N.a_1 a_2 \dots a_n, \dots, \quad (5)$$

at margin of error  $10^{-n}$ . The  $g$ -limit of (3) is nonterminating decimal (5) provided the  $n$ th digits are not all 0 beyond a certain value of  $n$ ; otherwise, it is terminating. As in standard analysis where a sequence converges, i.e., tends to a specific number, in the standard norm, a standard  $g$ -sequence, converges to its  $g$ -limit in the  $g$ -norm where the  $g$ -norm of a decimal is itself.

We define the  $n$ th distance  $d_n$  between two decimals  $a, b$  as the numerical value of the difference between their  $n$ th  $g$ -terms,  $a_n, b_n$ , i.e.,  $d_n = |a_n - b_n|$  and their  $g$ -

distance is the g-limit of  $d_n$ , We denote by  $\mathbf{R}^*$  the g-closure of  $\mathbf{R}$ , i.e., its closure in the g-norm.

A terminating decimal is a degenerate nonterminating decimal, i.e., the digits are all 0 beyond a certain value of  $n$ . The  $n$ th g-term of a nonterminating decimal repeats every preceding digit in the same order so that if finite terms are deleted or altered the  $n$ th g-term and, therefore, also the g-limit is unaltered as the remaining terms generate its g-sequence. Thus, a nonterminating decimal may have many g-sequences and we consider them equivalent for having the same g-limit.

Since addition and multiplication and their inverse operations subtraction and division are defined only on terminating decimals computing nonterminating decimals is done by approximation each by its  $n$ th g-terms (called  $n$ -truncation) and using their approximation to find the  $n$ th g-term of the result as its approximation at the same margin of error. This is standard computation, i.e., approximation by decimal segment at the  $n$ th digit. Thus, with our premises we have retained standard computation but avoided the contradictions and paradoxes of the real numbers. We have also avoided vacuous statement, e.g., vacuous approximation, because nonterminating decimals are g-limits of g-sequences which belong to  $\mathbf{R}^*$ . Moreover, we have contained the inherent ambiguity of nonterminating decimals by approximating them by their  $n$ th g-terms which are not ambiguous being terminating decimals. In fact, the ambiguity of  $\mathbf{R}^*$  has been contained altogether by its construction on the additive and multiplicative identities 0 and 1.

As we raise  $n$ , the tail digits of the  $n$ th g-term of any decimal recedes to the right indefinitely, i.e., it becomes steadily smaller until it is unidentifiable. While it tends to 0 in the standard norm it never reaches 0 and is not a decimal since its digits are not fixed; ultimately, they are indistinguishable from the similarly receding tail digits of the other nonterminating decimals. In iterated computation when we are trying to get closer and closer approximation of a decimal, e.g., calculating  $f(n) = (2n^4+1)/3n^4$ ,  $n = 1, 2, \dots$ , the tail digits may vary but recede to the right indefinitely and become steadily smaller leaving fixed digits behind that define a decimal. We approximate the result by taking its initial segment, the  $n$ th g-term, to desired margin of error.

Consider the sequence of decimals,

$$(\delta)^n a_1 a_2 \dots a_k, n = 1, 2, \dots, \quad (6)$$

where  $\delta$  is any of the decimals, 0.1, 0.2, 0.3, ..., 0.9,  $a_1, \dots, a_k$ , basic integers (not all 0 simultaneously). We call the nonstandard sequence (6) d-sequence and its  $n$ th term  $n$ th d-term. For fixed combination of  $\delta$  and the  $a_j$ 's,  $j = 1, \dots, k$ , in (6) the  $n$ th term is a terminating decimal and as  $n$  increases indefinitely it traces the tail digits of some nonterminating decimal and becomes smaller and smaller until we cannot see it anymore and indistinguishable from the tail digits of the other decimals (note that the  $n$ th d-term recedes to the right with increasing  $n$  by one decimal digit at a time). The sequence (6) is called nonstandard d-sequence since the  $n$ th term is not standard g-term; while it has standard limit (in the standard norm) which is 0 it is not a g-limit since it is not a decimal but it exists because it is well-defined by its nonstandard d-sequence. We call its nonstandard g-limit dark number and denote by  $d$ . Then we call its norm d-norm (distance from 0) which is  $d > 0$ . Moreover, as the  $n$ th term becomes smaller and smaller with indefinitely increasing  $n$  it is greater than 0 no matter how large  $n$  is so that if  $x$  is a decimal,  $0 < d < x$ .

Now, we allow  $\delta$  to vary steadily in its domain and also the  $a_j$ s along the basic integers (not simultaneously 0). Then their terms trace the tail digits of all the decimals and as  $n$  increases indefinitely they become smaller and smaller and

indistinguishable from each other. We call their nonstandard limits dark numbers and denote by  $d^*$  which is set valued and countably infinite and includes every g-limit of the nonstandard d-sequence (6). To the extent that they are indistinguishable  $d^*$  is a continuum (in the algebraic sense since no notion of open set is involved). Thus, the tail digits of the nonterminating decimals merge and form the continuum  $d^*$ .

At the same time, since the tail digits of all the nonterminating decimals form a countable combination of the basic digits 0, 1, ..., 9 they are countably infinite, i.e., in one-one correspondence with the integers. In fact, any set that can be labeled by integers or there is some scheme for labeling them by integers is in one-one correspondence with the integers, i.e., countably infinite. It follows that the countable union of countable set is countable. Therefore, the decimals and their tail digits are countably infinite. However, as the nth d-terms of (6) trace the tail digits of the nonterminating decimals they become unidentifiable and cannot be labeled by the integers anymore; therefore, they are no longer countable. In fact they merge as the continuum  $d^*$ .

Like a nonterminating decimal, an element of  $d^*$  is unaltered if finite g-terms are altered or deleted from its g-sequence. When  $\delta = 1$  and  $a_1 a_2 \dots a_k = 1$  (6) is called the basic or principal d-sequence of  $d^*$ , its g-limit the basic element of  $d^*$ ; basic because all its d-sequences can be derived from it. The principal d-sequence of  $d^*$  is,

$$(0.1)^n, n = 1, 2, \dots \tag{7}$$

obtained by the iterated difference,

$$\begin{aligned} N - (N - 1).99\dots &= 1 - 0.99\dots = 0 \text{ with excess} \\ &\quad \text{remainder of } 0.1; \\ 0.1 - 0.09 &= 0 \text{ with excess} \\ &\quad \text{remainder of } 0.01; \\ 0.01 - 0.009 &= 0 \text{ with excess} \\ &\quad \text{remainder of } 0.001; \\ &\quad \dots\dots \end{aligned} \tag{8}$$

Taking the nonstandard g-limits of the left side of (8) and recalling that the g-limit of a decimal is itself and denoting by  $d_p$  the d-limit of the principal d-sequence on the right side we have,

$$N - (N - 1).99\dots = 1 - 0.99\dots = d_p. \tag{9}$$

Since all the elements of  $d^*$  share its properties then whenever we have a statement “an element d of  $d^*$  has property P” we may write “ $d^*$  has property P”, meaning, this statement is true of every element of  $d^*$ . This applies to any equation involving an element of  $d^*$ . Therefore, we have,

$$d^* = N - (N - 1).99\dots = 1 - 0.99\dots \tag{10}$$

Like a decimal, we define the d-norm of  $d^*$  as  $d^* > 0$ .

**Theorem.** The d-limits of the indefinitely receding (to the right) nth d-terms of  $d^*$  is a continuum that coincides with the g-limits of the tail digits of the nonterminating decimals traced by those nth d-terms as the  $a_k$ s vary along the basic digits.

If  $x$  is nonzero decimal, terminating or nonterminating, there is no difference between  $(0.1)^n$  and  $x(0.1)^n$  as they become indistinguishably small as  $n$  increases indefinitely. This is analogous to the sandwich theorem of calculus that says,  $\lim(x/\sin x) = 1$ , as  $x \rightarrow 0$ ; in the proof, it uses the fact that  $\sin x < x < \tan x$  or  $1 < x/\sin x < \sec x$  where both extremes tend to 1 so that the middle term tends to 1 also. In our case, if  $0 < x < 1$ ,  $0 < x(0.1)^n < (0.1)^n$  and both extremes tend to 0 so must the middle term and they become indistinguishably small as  $n$  increases indefinitely. If  $x > 1$ , we simply reverse the inequality and get the same conclusion. Therefore, we may write,  $xd_p = d_p$  (where  $d_p$  is the principal element of  $d^*$ ), and since the elements of  $d^*$  share this property we may write  $xd^* = d^*$ , meaning, that  $xd = d$  for every element  $d$  of  $d^*$ . We consider  $d^*$  the equivalence class of its elements. In the case of  $x + (0.1)^n$  and  $x$ , we look at the  $n$ th  $g$ -terms of each and, as  $n$  increases indefinitely,  $x + (0.1)^n$  and  $x$  become indistinguishable. Now, since  $(0,1)^n > ((0.1)^n)^n > 0$  and the extreme terms both tend to 0 as  $n$  increases indefinitely, so must the middle term tend to 0 so that they become indistinguishably small (the reason  $d^*$  is called dark for being indistinguishable from 0 yet greater than 0): We summarize our discussion as follows:

$$\begin{aligned} &\text{if } x \text{ is not a new integer, } x + d^* = x; \\ &\quad \text{otherwise, if } x = N.99\dots \quad x + d^* \\ &\quad = N+1, \quad x - d^* = x; \\ &\text{if } x \neq 0, \quad xd^* = d^*; \quad (d^*)^n \\ &\quad = d^*, \quad n = 1, 2, \dots, \quad N = 0, 1, \dots \end{aligned} \quad (11)$$

$$\begin{aligned} 1 - d^* &= 0.99\dots, \quad N - (N - 1).99\dots \quad 1 - 0.99\dots \\ &= d^*, \quad N = 1, 2, \dots \end{aligned} \quad (12)$$

It follows that the  $g$ -closure of  $\mathbf{R}$ , i.e., its closure in the  $g$ -norm, is  $\mathbf{R}^*$  which includes the additive and multiplicative inverses and  $d^*$ . We also include in  $\mathbf{R}^*$  the upper bounds of the divergent sequences of terminating decimals and integers (a sequence is divergent if the  $n$ th terms are unbounded as  $n$  increases indefinitely, e.g., the sequence 9, 99, ...) called unbounded number  $u^*$  which is countably infinite since the set of sequences is. We follow the same convention for  $u^*$ : whenever we have a statement “ $u$  has property  $P$  for every element  $u$  of  $u^*$ ” we can simply say “ $u^*$  has property  $P$ ”. Then  $u^*$  satisfies these dual properties:

$$\text{for all } x, \quad x + u^* = u^*; \quad \text{for } x \neq 0, \quad xu^* = u^*. \quad (13)$$

Neither  $d^*$  nor  $u^*$  is a decimal and their properties are solely determined by their sequences. Then  $d^*$  and  $u^*$  have the following dual or reciprocal properties and relationship:

$$\begin{aligned} 0d^* &= 0, \quad 0/d^* = 0, \quad 0u^* = 0, \quad 0/u^* \\ &= 0, \quad 1/d^* = u^*, \quad 1/u^* = d^*. \end{aligned} \quad (14)$$

Numbers like  $u^* - u^*$ ,  $d^*/d^*$  and  $u^*/u^*$  are still indeterminate but indeterminacy is avoided by computation with the  $g$ - or  $d$ -terms.

The decimals are linearly ordered by the lexicographic ordering “ $<$ ” defined as follows: two elements of  $\mathbf{R}$  are equal if corresponding digits are equal. Let

$$N.a_1a_2\dots, \quad M.b_1b_2\dots \in \mathbf{R}. \quad (15)$$

Then,

$$N.a_1a_2. \dots < M.b_1b_2 \text{ if } N < M \text{ or if } N = M, a_1 < b_1; \text{ if } a_1 = b_1, a < b_2; \dots, \quad (16)$$

and, if  $x$  is any decimal we have,

$$0 < d^* < x < u^* \quad (17)$$

The trichotomy axiom follows from lexicographic ordering. This is the natural ordering mathematicians sought among the real numbers but it does not exist there because it contradicts the trichotomy axiom.

### 1.3. Duals and their reciprocals

Mathematical systems are better understood by bringing in the notion of dual systems because it introduces some symmetry that may be useful. We can look at divergent sequences, i.e., sequences whose terms become bigger and bigger that we can no longer comprehend them and become indistinguishable from each other, as dual of convergent sequences. In this sense the divergent sequences also form a continuum. We denote their upper bounds by  $u^*$  which satisfies (12) and (15). Then we look at  $d^*$  as the dual of  $u^*$  and  $\mathbf{R}^*$  that of the system of additive and multiplicative inverses (which has holes, namely, the nonexistent multiplicative inverses of integers). Thus  $\mathbf{R}^*$  is a semi-field, the nonzero integers forming a semi-ring since some of them have no multiplicative inverses. Like  $d^*$ ,  $u^*$  cannot be separated from the decimals, i.e., there is no boundary between either of them and the decimals and between finite and infinite, i.e., we cannot separate  $d^*$  from a decimals and there is no boundary to cross between finite and infinite so that beyond a certain finite decimal everything else is infinite. The latter is what is meant by the expression  $u^* + x = u^*$  for any decimal  $x$ . Duality is also seen in this case: Let  $\lambda > 1$  be terminating decimal then the sequence  $\lambda^n$ ,  $n = 1, 2, \dots$ , diverges to  $u^*$  but  $(1/\lambda)^n$ ,  $n = 1, 2, \dots$  converges,  $d\text{-lim } (1/\lambda)^n = d^*$ .

### 1.4 Isomorphism between the integers and the decimal integers

To find out more about the structure of  $\mathbf{R}^*$  we show the isomorphism between the integers and the decimal integers, i.e., integers of the form,

$$N.99\dots, N = 0, 1, \dots \quad (18)$$

but before doing so we first note that  $1 + 0.99\dots$  is not defined in  $\mathbf{R}$  since  $0.99\dots$  is nonterminating but we can write  $0.99\dots = 1 - d^*$  so that  $1 + 0.99\dots = 1 + 1 - d^* = 2 - d^* = 1.99\dots$  and we now define  $1 + 0.99\dots = 1.99\dots$  or, in general,  $N - d^* = (N-1).99\dots$ . The pairs  $(N, (N-1).99\dots)$ ,  $N = 1, 2, \dots$ , are called twin integers because they are isomorphic:

Let  $f$  be the mapping  $N \rightarrow (N-1).99\dots$  then we show that  $f$  is an isomorphism between the integers and decimal integers. This means that addition and multiplications of integers is the same as addition and multiplication of decimal integers. We include in this isomorphism the map  $d^* \rightarrow 0$ , so that its kernel is the set  $\{d^*, 0.99\dots\}$  from which follows:

$$(d^*)^n = d^*; (0.99\dots)^n = 0.99\dots, n = 1, 2, \dots \quad (19)$$

(The second equation of (19) can be proved also by mathematical induction)

For the curious reader we exhibit other properties of  $0.99\dots$ . Let  $K$  be an integer,  $M.99\dots$  and  $N.99\dots$  decimal integers. Then

$$\begin{aligned} (a) \quad & K + M.99\dots = (K+M).99\dots \\ (b) \quad & K(M.99\dots) = K(M + 0.99\dots) \\ & = KM + K(0.99\dots) = KM + (K-1).99\dots \\ (c) \quad & M.99\dots + N.99\dots \\ & = M + N + 0.99\dots + 0.99\dots. \end{aligned} \quad (20)$$

$$(d) \quad (M.99\dots)(N.99\dots) = (MN+M+N-1).99\dots$$

$$(e) \quad 0.99\dots + 0.99\dots = 2(0.99\dots) = 1.99\dots \quad (21)$$

We extended the isomorphism to include  $d^*$  by the mapping  $f(0) = d^*$ , even if  $d^*$  is neither a decimal nor an integer, because  $d^*$  behaves like 0 and  $0.99\dots$  like 1. The isomorphism makes the decimal integers also integers (i.e., equivalent and behave alike).

### 1.5 Adjacent decimals and recurring 9s

Two decimals are adjacent if they differ by  $d^*$ . Predecessor-successor pairs and twin integers are adjacent. In particular,  $74.5700\dots$  and  $74.5699\dots$  are adjacent.

Since the decimals have the form  $N.a_1a_2\dots a_n\dots$ ,  $N = 0, 1, 2, \dots$ , the digits are identifiable and, in fact, countably infinite and they are linearly ordered by lexicographic ordering. Therefore, they are discrete or digital and the adjacent pairs are also countably infinite. However, since their tail digits form a continuum,  $\mathbf{R}^*$  is a continuum with the decimals its countably infinite discrete subsystem.

A decimal is called recurring 9 if its tail decimal digits are all equal to 9. For example,  $4.3299\dots$  and  $299.99\dots$  are recurring 9s; so are the decimal integers. (In an isomorphism between two algebraic systems, their operations are interchangeable, i.e., they have the same algebraic structure but differ only in notation).

The recurring 9s have interesting properties. For instance, the difference between the integer  $N$  and the recurring 9,  $(N - 1).99\dots$ , is  $d^*$ ; such pair of decimals are called adjacent because there is no decimal between them and they differ by  $d^*$ . In the lexicographic ordering the smaller of the pair of adjacent decimals is the predecessor and the larger the successor. The average between them is the predecessor. Thus, the average between 1 and  $0.99\dots$  is  $0.99\dots$  since  $(1.99\dots)/2 = 0.99\dots$ ; this is true of any recurring 9, say,  $34.5799\dots$  whose successor is  $34.5800\dots$ . Conversely, the g-limit of the iterated or successive averages between a fixed decimal and another decimal of the same integral part is the predecessor of the former.

Since adjacent decimals differ by  $d^*$  and there is no decimal between them, i.e., we cannot split  $d^*$  into nonempty disjoint sets, we have another proof that  $d^*$  is a continuum (in the algebraic sense). Then we have another proof that  $\mathbf{R}^*$  is a continuum (also in the algebraic sense).

It follows from the counterexample to the trichotomy axiom that an irrational number cannot be expressed as limit of sequence of rationals since the closest it can get to it is some rational interval which still contains some rational whose relationship to it is unknown.

Now we know what the eurrationalals are; they are the nonterminating decimals, periodic and nonperiodic. The g-sequence of an eurrational, which is a sequence of rationals, gets directly to its g-limit, digit by digit. We note further that an eurrational is an infinite series in terms of its digits as follows:

$$N.a_1a_2\dots a_n\dots = N + .a_1 + .0a_2 + \dots + .00\dots 0a_n + \dots; 0.99\dots = 0.9 + 0.09 + \dots \quad (21)$$

## 1.6 The structure of $\mathbf{R}^*$ and its subspaces

We add the following results to the information we now have about the various subspaces of  $\mathbf{R}^*$  to provide a full picture of the structure of the new real number system. The next theorem is a definitive result about the continuum  $\mathbf{R}^*$ .

**Theorem.** In the lexicographic ordering  $\mathbf{R}^*$  consists of adjacent predecessor-successor pairs (each joined by  $d^*$ ); therefore, the g-closure  $\mathbf{R}^*$  of  $\mathbf{R}$  is a continuum.

However, the decimals form countably infinite discrete subspace of  $\mathbf{R}^*$  since there is a scheme for labeling them by the integers.

We can imagine the terminating decimals as forming a right triangle with one edge horizontal and the vertical one extending without bounds. The integral parts are lined up on the vertical edge and they are joined together by their branching digits between the hypotenuse and the horizontal and extend to  $d^*$  which is adjacent to 0 (differs from 0 by a dark number) at the vertex of the horizontal edge.

**Corollary.**  $\mathbf{R}^*$  is non-Archimedean and non-Hausdorff in both the standard and the g-norm and the subspace of decimals are countably infinite, hence, discrete but Archimedean and Hausdorff.

The following theorem is standard in the real in the real number system with the standard norm. Therefore, we do not bring in  $d^*$  in the proof so that this is really a theorem about the decimals with the standard norm which is not true in the g-norm because the decimals merge into a continuum at their tail digits and cannot be separated.

**Theorem.** The rationals and irrationals are separated, i.e., they are not dense in their union (this is the first indication of discreteness of the decimals).

This means that every decimal is separated from the rest, the terminating decimals from the eurrationalals and from each other.

The next theorem has standard proof (in  $\mathbf{R}$ ); it raised eyebrows in internet forums.

**Theorem.** The largest and smallest elements of the open interval (0,1) are  $0.99\dots$  and  $1 - 0.99\dots$ , respectively.

The next theorem used to be called Goldbach's conjecture; has now a proof in  $\mathbf{R}^*$ .

**Theorem.** An even number greater than 2 is the sum of two prime numbers.

This is unsolved because, like Fermat's equation (FLT), it is indeterminate. Before proving the theorem, we first note that an integer is a prime if it leaves a positive remainder when divided by another integer other than 1. We retain this definition in  $\mathbf{R}^*$ .

We now have a sense of how the decimals are arranged by the lexicographic ordering. Consider the decimals with integral part N:

$$\begin{array}{l}
N.99\dots\dots\dots \\
\dots\dots\dots \\
N.4800\dots\dots\dots \\
N.4799\dots\dots\dots \\
\dots\dots\dots \\
N.10\dots\dots\dots \\
\dots\dots\dots \\
N.00\dots0100\dots \\
\dots\dots\dots \\
\\
N.00\dots \qquad \qquad \qquad (22)
\end{array}$$

The largest decimal in the set is the decimal integer N.99... and the smallest is the terminating decimal N.00... From the bottom up the decimals with integral part N are arranged as predecessor-successor pairs each joined by d\*. Each gap indicated by the ellipses is filled by countably infinite adjacent predecessor-successor pairs each also joined by d\* so that their union is a continuum. We now have a clear picture of how  $\mathbf{R}^*$  is arranged on the new real line linearly ordered by  $<$ , the lexicographic ordering.

**A. Important results; resolution of a paradox**

- (1) Every convergent sequence has a g-subsequence that defines a decimal adjacent to its limit. If the decimal is terminating it is the limit itself.
- (2) It follows from (1) that the limit of a sequence of terminating decimals can be found by evaluating the g-limit of its g-subsequence which is adjacent to it. We can use this as alternative way of computing the limit of ordinary sequence.
- (3) In Horgan, H., The death of proof, Scientific American, 1993, several counterexamples to the generalized Jourdan curve theorem for n-sphere are shown where a continuous curve has points in both the interior and exterior of the n-sphere,  $n = 2, 3, \dots$ , without crossing the n-sphere. The explanation is: the functions cross the n-sphere through dark numbers.
- (5) Given two decimals and their g-sequences and respective nth g-terms  $A_n, B_n$  we define nth g-distance as the g-norm  $|A_n - B_n|$  of the difference between their nth g-terms. Then their g-distance is the  $g\text{-lim } |A_n - B_n|$ , as  $n \rightarrow \infty$ , which is adjacent to the standard norm of the difference. The advantage here is that the g-distance is the g-norm of their decimal difference and the difference between nonterminating decimals cannot be evaluated otherwise. Moreover, this notion of distance can be extended to n-space,  $n = 2, 3, \dots$ , and the distance between two points can be evaluated digit by digit in terms of their components without the need for evaluating roots. In fact, any computation in the g-norm yields the results directly, digit by digit, without the need for intermediate computation such as evaluation of roots in standard computation.

**B. More on nonstandard numbers**

We highlight some properties of special class of nonstandard numbers that can be checked by looking at their g- or divergent sequences. The principal element of d\* (g-limit of its principal g-sequence) is dark number of order 0.1.

Let  $\gamma$  be a fraction such that  $0 < \gamma < 1$  and let  $d_\gamma = g\text{-lim } \gamma^n$ , as  $n \rightarrow \infty$ , n integer,  $d_\gamma$  is called dark number of order  $\gamma$ . An unbounded number u of order  $\lambda > 1$  is defined as the upper bound of the sequence  $\lambda^n$ , as  $n \rightarrow \infty$ . The number u is an element of  $u^*$  just

as  $d_\gamma$  is an element of  $d^*$ . Since  $\gamma^n$  is positive and steadily decreasing,  $d_\gamma$  is less than any given decimal. (In this section we only consider positive decimal and hence we shall drop the qualification positive) To see this, let  $x$  be any decimal; since  $\gamma < 1$ , the integer  $n$  can be chosen large enough that  $0 < d_\gamma < \lambda^n < x$ . Similarly, it can be an unbounded decimal of any order greater than any decimal.

The following is obvious by checking their  $g$ -sequences.

1. The product of any decimal and dark number of order  $\gamma$  is dark number of order  $\gamma$ ; the product of a decimal and unbounded number of order  $\lambda$  is unbounded of order  $\lambda$ .

2. If  $d_1$  and  $d_2$  are dark numbers of order  $\gamma_1$  and  $\gamma_2$ , respectively, where  $\gamma_1 < \gamma_2$ , then  $d_1 + d_2$  is dark number of order  $\gamma_2$ ,  $d_2 - d_1$  is dark number of order  $\gamma_1$ ,  $d_1/d_2$  is dark number of order  $\gamma_1/\gamma_2$  and  $d_1/d_2$  is unbounded number of order  $\gamma_2/\gamma_1$ .

3. A decimal divided by a dark number of order  $\gamma$  is unbounded of order  $1/\gamma$ ; a decimal divided by unbounded number of order  $\lambda$  is a dark number of order  $1/\lambda$ ; the reciprocal of dark number of order  $\gamma$  is unbounded number of order  $1/\gamma$ ; the reciprocal of unbounded number of order  $\lambda$  is dark number of order  $1/\lambda$ .

4. If  $\mu_1, \mu_2$  are unbounded numbers of orders  $\lambda_1, \lambda_2$ , respectively, where  $\lambda_1 > \lambda_2$ , then  $\mu_1 + \mu_2$  and  $\mu_1 - \mu_2$  are both unbounded numbers of order  $\lambda_1$  and  $\mu_1/\mu_2$  and  $\mu_2/\mu_1$  are unbounded and dark numbers of orders  $\lambda_1/\lambda_2$  and  $\lambda_2/\lambda_1$ , respectively.

5. The sum of two dark numbers of the same order is a dark number of that order; the quotient of two dark numbers is indeterminate but can be avoided using  $n$ th  $g$ -term approximation. If the  $n$ th  $g$ -term of the quotient is a decimal then the quotient is a decimal, if it is greater than 1 the quotient is  $u^*$ ; if it is less than 1 the quotient is  $d^*$ .

These results, are useful in avoiding indeterminate forms in calculation.

Moreover, since all elements of  $d^*$  share the properties of  $d^*$ , we can use any element of this class for our argument in proving a theorem, especially, in dealing with inequality, the advantage being that it has clear structure. Consequently, there is no loss of generality in using the principal  $n$ th  $g$ -term of  $d^*$  for any purpose involving  $d^*$ .

**Remark.** Gauss' diagonal method proves neither the existence of nondenumerable set nor a continuum; it proves only the existence of countably infinite set, i.e., the off-diagonal elements consisting of countable union of countably infinite sets. The off-diagonal elements are not even well-defined because we know nothing about their digits (a decimal is determined by its digits). Therefore, we raise these conjectures:

**Conjectures.** (1) Nondenumerable set does not exist; (2) Only discrete set has cardinality; a continuum has none.

### 1.7 The counterexamples to FLT

Given the contradiction in negative statement, we use Fermat's equation in place of the statement of FLT so that its solutions are counterexamples to FLT. We first summarize the properties of the basic digit 9.

(1) String of 9s differs from nearest power of 10 by 1, e.g.,  $10^{100} - 99\dots9 = 1$ .

(2) If  $N$  is an integer, then  $(0.99\dots)^N = 0.99\dots$  and, naturally, both sides of the equation have the same  $g$ -sequence. Therefore, for any integer  $N$ ,  $((0.99\dots)10)^N = (9.99\dots)10^N$ .

(3)  $(d^*)^N = d^*$ ;  $((0.99\dots)10)^N + d^* = 10^N$ ,  $N = 1, 2, \dots$

Then the exact solutions of Fermat's equation are given by the triple  $(x, y, z) = ((0.99\dots)10^T, d^*, 10^T)$ ,  $T = 1, 2, \dots$ , that clearly satisfies Fermat's equation,

$$x^n + y^n = z^n, \quad (23)$$

for  $n = NT > 2$ . Moreover, for  $k = 1, 2, \dots$ , the triple  $(kx, ky, kz)$  also satisfies Fermat's equation. They are the countably infinite counterexamples to FLT that prove the conjecture false. One counterexample is, of course, sufficient to disprove a conjecture.

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