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The New Real Number System

Abstract: The paper points out the inconsistency and ambiguity of the field axioms of the real number system and notes that the only clearly defined and consistent mathematical model of the real numbers is the set of terminating decimals. Then it identifies present mathematics having global applications. They are continuous and discrete; the former meets the needs of the natural sciences since physical space is a continuum that pervades everything in nature and the latter of computing and applications since physical systems are discrete. Then the base mathematical space over which mathematics is to be built called the new real number system is developed using three consistent axioms. This new mathematical space is a continuum, non-Archimedean and non-Hausdorff but its subspace of decimals is discrete, Archimedean and Hausdorff. It introduces a new norm that has many advantages over the other norms of the real number system, especially, for purposes of computing.

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1. INTRODUCTION

Why do we need a new real number system? The real number system defined by the field axioms which is supposedly a complete ordered field has a number of defects. It is neither complete nor ordered nor a field. We make the following observation.

(1) The counterexample to the trichotomy axiom constructed by Felix Brouwer reveals the field axioms are inconsistent. An inconsistent mathematical system collapses since a theorem derived from some axioms is contradicted by another. Since the book that presents Brouwer's counterexample to the trichotomy axiom is not readily available the author presents below a different version of the counterexample to the trichotomy axiom that shows at the same time that the terminating decimals or fractions are not linearly ordered by "<" and the irrationals ill-defined.

(2) Linear ordering of the real numbers is necessary to put the real numbers on the real line for the purposes of analysis. All along we have assumed that the real numbers linear ordering of the reals which is not true.

(3) The only consistent and clearly defined number system with global application is the system of terminating decimals but we do not know the right axioms for them that retain all the interesting properties of the real numbers and that is what we shall find out here.

(4) The real numbers are assumed to be infinite; an infinite set is inherently ambiguous since not all its elements are known, identifiable or computable. Therefore, any categorical statement about its elements such as one involving the universal quantifier "every" or the existential quantifier "there exists" is unverifiable.

(5) Large and small numbers are ambiguous due to our limited capability to compute them (present technology allows computation of limited number of digits of a number at a time).

(6) Most theorems of the real numbers are proved from the definition and properties of the decimals. When, occasionally, we use a field axiom to prove a theorem, e.g., the axiom of choice in the proof of existence of nonmeasurable set, we are confronted with such contradiction as the Banach-Tarski paradox]. This paradox is really due to the use of the existential quantifier "there exists" in the proof of the existence of nonmeasurable set that plays the major role in arriving at it and the axiom of choice is quite incidental.

(7) The real number system has lots of ambiguity which is a source of contradiction; among the sources of ambiguity aside from those we have already noted are:

(a) Vacuous concept or proposition (declarative statement); e.g., "Let N be the largest integer". This statement leads to the Perron paradox that says: The largest integer is 1. To avoid this situation we put at the outset all the concepts we need in the construction of a mathematical space; then we do not need to prove existence later.

(b) Ill-defined concept (negation of well-defined) or proposition. A concept is well-defined if its existence, properties and relationship with other concepts are specified by the axioms (note that existence is important to avoid vacuous proposition). A proposition is ill-defined if it involves an ill-defined concept. In particular, the use of undefined concept is inadmissible as it introduces unnecessary ambiguity.

(c) Self-referent proposition, i.e., the conclusion refers to the hypothesis. All the Russell Paradoxes belong to this category, e.g., the barber paradox: The barber of Seville shaves those and only those who do not shave themselves; who shaves the barber? Incidentally, the indirect proof is a case of self-reference and the only way to avoid it is to stick to constructivist mathematics of which this paper is an example.

(e) Invalid extension. The axioms of a mathematical space do not apply to any of its extensions since the latter lies in its complement; therefore, each extension requires new concepts and separate axioms consistent with the initial axioms. Improper extension is the source of the problem with the imaginary number i from which we can derive the contradiction, 1 = 0.

(f) The use of universal rules of inference such as formal logic that has nothing to do with the axioms of the given mathematical space. The latter must be completely defined by its axioms including its rules of inference or logic; they must be specific to it.

(g) Unclarity on the subject matter of the given mathematical space. It cannot be the concepts of individual thought since they are inaccessible to others, cannot be studied collectively and, therefore, cannot be axiomatized as a mathematical space. David Hilbert provided the remedy by requiring the subject matter of mathematics to be the representation of thought by symbols that everyone can examine (we also call them concepts) subject to consistent axioms.

We now present our counterexample to the trichotomy axiom. Let C be an irrational number. We want to isolate C in an interval such that all the decimals to the left of C are less than C and all decimals to the right of C are greater than C. We do this by constructing a sequence of smaller and smaller rational intervals (rational endpoints) such that each interval in the sequence is inside the preceding (this is called nested sequence of intervals). In the construction we skip the rationals that do not satisfy the above condition. Although given two distinct rationals x, y we can tell if $x < y$ or $x > y$, we cannot line them up on the real line under the relation “<”since if x, y are two rationals, x < y, there is an infinity of rationals between them and we cannot verify their arrangement. Therefore, we settle for this scenario: starting with the rational interval [A,B] we find a nested sequence of rational intervals that “insures” C lies between the two endpoints at each stage. We go for an arrangement that will allow us to distinguish the left from the right endpoints of the sequence. We construct rows of rationals starting with numerator 1 in the first row, 2 in the second row, etc., and the denominators in each case consisting of consecutive integers starting from 1 in increasing order going left so that in each case we start with a denominator of a potential right endpoint. Actually, we can squeeze the rows into a single row since no particular order with respect to “<” is involved.

Even this arrangement is a problem. For example, suppose at a certain stage in the construction we have a right endpoint 1/5 then the number 20/100 appears on the left. Then, in trying to pair the right endpoint 1/5 with a left endpoint we skip 20/100 and all other rationals to the right of 1/5 in the ordering “<” that appear on the left and move further left than all of them. We do the same in choosing the right endpoints moving inward. Without loss of generality, we take this rational 1/5 to be the first right endpoint in the construction. Then once we have found the left pair for 1/5 we either use this as the left endpoint of the next rational interval and pair it with some rational on the left of 1/5 or find a new left endpoint to the right of the first left endpoint to pair with a right endpoint left of 1/5, etc. We make sure that we do not get closer to C than 10^{-n} at the nth step in the choice of the first n endpoints so that C remains inside each interval. While we are sure for all left and right endpoints A, B that we have already identified in our construction, A < C < B and all rationals right of A and left of B in the ordering “<” satisfy this inequality, there remains an interval of rational endpoints containing C and rationals that do not satisfy this inequality no matter how large we choose n. Therefore, the location of C remains unknown.

This construction attests to the ambiguity of the concept *irrational* and the problem of representing an irrational as limit of a sequence of rationals; for every such sequence there is always a gap. As we have just seen even the rationals are ambiguous mainly because there is an infinity of rationals between any given two rationals so that we cannot order them under the relation “<”, i.e., we cannot line them up in the line interval between 0 and 1, denoted by [0,1], in accordance with this relation. This is due to the ambiguity of infinity. Consequently, the real number system has no ordering under this relation and the trichotomy axiom that says, given two real numbers x, y, one and only one of the following holds: x < y, x = y, x > y, is false.

This construction shows the fractions just as ambiguous as the nonterminating decimals the latter having a bit of advantage for being linearly ordered by the lexicographic ordering (see below). We shall see later that the new real numbers are linearly ordered by “<” under the lexicographic ordering and satisfy the trichotomy axiom.

2. OUR STRATEGY

Our strategy is not simply to build a contradiction-free mathematical space that we shall call the new real number system **R*** but also to meet the needs of the natural sciences and practical affairs. This means that the new real number system must contain mathematics that has global applications. In particular, it must provide both continuous and discrete mathematics including the decimals whose physical model, the metric system, has global applications that other systems of measures are converting to it. Any other useful mathematics that may arise we shall consider a bonus. Concretely, the new real number system must be a continuum since physical space is which pervades everything and cannot be split into disjoint nonempty subsets. The key is to choose the right consistent axioms upon which to build the new real number system. These are the parameters for our construction.

3. THE TERMINATING DECIMALS

We first build the terminating decimals **R** as our base space subject to the following:

Axiom 1. 0 and 1 are elements of **R**.
Axioms 2 – 3. The addition and. multiplication tables that well-define 0, 1, the integers and the terminating decimals.

The elements 0 and 1 are called the additive and multiplicative identities. For the moment they are not well-defined but they will be by the addition and multiplication tables. Moreover, **R** is completely defined by the axioms and, naturally, mathematical reasoning is solely determined by them.

We first define the digits or basic integers beyond 0 and 1:

$$1 + 1 = 2;\ 2 + 1 = 3;\ \ldots,\ 8 + 1 = 9. \quad (1)$$

(We omit the statement of the addition and multiplication tables which is familiar to everyone since primary school) Then we define the rest of the integers as base 10 place-value numerals:

$$a_n a_{n-1} \ldots a_1 = a_n 10^n + a_{n-1} 10^{n-1} + \ldots + a_1, \quad (2)$$

where the a_ns are basic integers.

Now, we extend the integers to include the additive and multiplicative inverses −x and, if x is not 0, 1/x (reciprocal of x), respectively. Note that the reciprocal of a non-zero integer exists only if it has no prime factor other than 2 or 5. This extension requires corresponding extension of the operations + and ×, in effect, re-stating associativity, etc., as part of its axioms and something else that is new: the rules of sign that we take as part of the axioms of this extension (we need not write them as they are familiar). Then we define a new operation: division of an integer x by a nonzero integer y, or quotient, denoted by x/y and defined by:

$$x/y = x(1/y). \quad (3)$$

This quotient exists when y has no prime factor other than 2 or 5. We similarly extend associativity and commutativity of addition and multiplication and distributivity of multiplication relative to addition and include them the axioms of the extension. We consider subtraction the inverse operation of addition and division that of multiplication, examples of duality that we shall consider in detail below.

We define a terminating decimal as follows:

$$\begin{aligned} a_n a_{n-1} \ldots a_1 . b_k b_{k-1} \ldots b_1 &= a_n 10^n + a_{n-1} 10^{n-1} + \ldots + a_1 + b_1/10 + b_2/10^2 + \ldots + b_k/10^k \\ &= a_n 10^n + a_{n-1} 10^{n-1} + \ldots + a_1 + b_1(0.1) + b_2(0.1)^2 + \ldots + b_k(0.1)^k. \end{aligned} \quad (4)$$

where $a_na_{n-1}\ldots a_1$ is the integral part, $b_1b_2\ldots b_k$ the decimal part and 0.1 = 1/10. Note that the terminating decimals are well-defined since the reciprocal of 10 has only the factors 2 and 5. Then the quotient x/y of an integer x by nonzero integer y exists only if y has no prime factor other than 2 or 5. Such quotient is called rational. We recall that in the real number system a rational is nonterminating periodic (a terminating decimal is periodic). This is ambiguous for nonterminating decimal since it is not verifiable.

This definition of the integers as the integral parts of the terminating decimals resolves the inadequacy of Peanos's postulates in the development of the natural numbers for they are clearly isomorphic to them and makes them integers.

4. THE NONTERMINATING DECIMALS

Now we define the nonterminating decimals for the first time without contradiction and with contained ambiguity, i.e., approximable by certainty. We build them on what we know: the terminating decimals, our point of reference for all its extensions.

A sequence of terminating decimals of the form,

$$N.a_1,\ N.a_1a_2,\ \ldots,\ N.a_1a_2\ldots a_n,\ \ldots \qquad (5)$$

where N is integer and the a_ns are basic integers, is called standard generating or g-sequence. Its nth g-term, $N.a_1a_2\ldots a_n$, defines and approximates its g-limit, the nonterminating decimal,

$$N.a_1a_2\ldots a_n,\ldots, \qquad (6)$$

at margin of error 10^{-n}. The g-limit of (5) is nonterminating decimal (6) provided the nth digits are not all 0 beyond a certain value of n; otherwise, it is terminating. As in standard analysis where a sequence converges, i.e., tends to a specific number, in the standard norm, a standard g-sequence, converges to its g-limit in the g-norm where the g-norm of a decimal is itself.

We define the nth distance d_n between two decimals a, b as the numerical value of the difference between their nth g-terms, a_n, b_n, i.e., $d_n = |a_n - b_n|$ and their g-distance is the g-limit of d_n, We denote by **R*** the g-closure of **R**, i.e., its closure in the g-norm.

A terminating decimal is a degenerate nonterminating decimal, i.e., the digits are all 0 beyond a certain value of n. The nth g-term of a nonterminating decimal repeats every preceding digit in the same order so that if finite terms are deleted or altered the nth g-term and, therefore, also the g-limit is unaltered as the remaining terms generate its g-sequence. Thus, a nonterminating decimal may have many g-sequences and we consider them equivalent for having the same g-limit.

Since addition and multiplication and their inverse operations subtraction and division are defined only on terminating decimals computing nonterminating decimals is done by approximation each by its nth g-terms (called n-truncation) and using their approximation to find the nth g-term of the result as its approximation at the same margin of error. This is standard computation, i.e., approximation by decimal segment at the nth digit. Thus, with our premises we have retained standard computation but avoided the contradictions and paradoxes of the real numbers. We have also avoided vacuous statement, e.g., vacuous approximation, because nonterminating decimals are g-limits of g-sequences which belong to **R***. Moreover, we have contained the inherent ambiguity of nonterminating decimals by approximating them by their nth g-terms which are not ambiguous being terminating decimals. In fact, the ambiguity of **R*** has been contained altogether by its construction on the additive and multiplicative identities 0 and 1.

As we raise n, the tail digits of the nth g-term of any decimal recedes to the right indefinitely, i.e., it becomes steadily smaller until it is unidentifiable. While it tends to 0 in the standard norm it never reaches 0 and is not a decimal since its digits are not fixed; ultimately, they are indistinguishable from the similarly receding tail digits of the other nonterminating decimals. In iterated computation when we are trying to get closer and closer approximation of a decimal, e.g., calculating $f(n) = (2n^4+1)/3n^4$, n = 1, 2, …, the tail digits may vary but recede to the right indefinitely and become steadily smaller leaving fixed digits behind that define a decimal. We approximate the result by taking its initial segment, the nth g-term, to desired margin of error.

Consider the sequence of decimals,

$(\delta)^n a_1 a_2 \ldots a_k$, n = 1, 2, …, (7)

where δ is any of the decimals, 0.1, 0.2, 0.3, …, 0.9, a_1, …, a_k, basic integers (not all 0 simultaneously). We call the nonstandard sequence (7) d-sequence and its nth term nth d-term. For fixed combination of δ and the a_j's, j = 1, …, k, in (7) the nth term is a terminating decimal and as n increases indefinitely it traces the tail digits of some nonterminating decimal and becomes smaller and smaller until we cannot see it anymore and indistinguishable from the tail digits of the other decimals (note that the nth d-term recedes to the right with increasing n by one decimal digit at a time). The sequence (7) is called nonstandard d-sequence since the nth term is not standard g-term; while it has standard limit (in the standard norm) which is 0 it is not a g-limit since it is not a decimal but it exists because it is well-defined by its nonstandard d-sequence. We call its nonstandard g-limit dark number and denote by d. Then we call its norm d-norm (standard distance from 0) which is d > 0. Moreover, while the nth term becomes smaller and smaller with indefinitely increasing n it is greater than 0 no matter how large n is so that if x is a decimal, 0 < d < x.

Now, we allow the a_js to vary along the basic integers (not simultaneously 0) as $n \to \infty$. Their terms trace the tail digits of all the decimals and as n increases indefinitely they steadily become smaller and indistinguishable from each other. We call their nonstandard limits dark numbers denoted by d* which is set valued and countably infinite and includes every g-limit of the nonstandard d-sequence (7). To the extent that they are indistinguishable d* is a continuum (in the algebraic sense since no notion of open set is involved). Thus, the tail digits of the nonterminating decimals merge and form the continuum d*.

At the same time, since the tail digits of all the nonterminating decimals form a countable combination of the basic digits 0, 1, …, 9 they are countably infinite, i.e., in one-one correspondence with the integers. In fact, any set that can be labeled by integers or there is some scheme for labeling them by integers is in one-one correspondence with the integers, i.e., countably infinite. It follows that the countable union of countable set is countable. Therefore, the decimals and their tail digits are countably infinite. However, as the nth d-terms of (7) trace the tail digits of the nonterminating decimals they become unidentifiable and cannot be labeled by the integers anymore; therefore, they are no longer countable. In fact they merge as the continuum d*.

Like a nonterminating decimal, an element of d* is unaltered if finite g-terms are altered or deleted from its g-sequence. When $\delta = 1$ and $a_1a_2\ldots a_k = 1$ (7) is called the basic or principal d-sequence of d*, its g-limit the basic element of d*; basic because all its d-sequences can be derived from it. The principal d-sequence of d* is,

$$(0.1)^n, \; n = 1, 2, \ldots \quad (8)$$

obtained by the iterated difference,

$$\begin{aligned} N - (N-1).99\ldots = 1 - 0.99\ldots &= 0 \text{ with excess remainder of } 0.1; \\ 0.1 - 0.09 &= 0 \text{ with excess remainder of } 0.01; \\ 0.01 - 0.009 &= 0 \text{ with excess remainder of } 0.001; \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \quad (9)$$

Taking the nonstandard g-limits of the left side of (9) and recalling that the g-limit of a decimal is itself and denoting by d_p the d-limit of the principal d-sequence on the right side we have,

$$N - (N-1).99\ldots = 1 - 0.99\ldots = d_p. \quad (10)$$

Since all the elements of d* share its properties then whenever we have a statement "an element d of d* has property P" we may write "d* has property P", meaning, this statement is true of every element of d*. This applies to any equation involving an element of d*. Therefore, we have,

$$d^* = N - (N-1).99\ldots = 1 - 0.99\ldots \quad (11)$$

Like a decimal, we define the d-norm of d* as d* > 0.

Theorem. The d-limits of the indefinitely receding (to the right) nth d-terms of d* is a continuum that coincides with the g-limits of the tail digits of the nonterminating decimals traced by those nth d-terms as the a_ks vary along the basic digits.

If x is nonzero decimal, terminating or nonterminating, there is no difference between $(0.1)^n$ and $x(0.1)^n$ as they become indistinguishably small as n increases indefinitely. This is analogous to the sandwich theorem of calculus that says, $\lim(x/\sin x) = 1$, as $x \to 0$; in the proof, it uses the fact that $\sin x < x < \tan x$ or $1 < x/\sin x < \sec x$ where both extremes tend to 1 so that the middle term tends to 1 also. In our case, if $0 < x < 1$, $0 < x(0.1)^n < (0.1)^n$ and both extremes tend to 0 so must the middle term and they become indistinguishably small as n increases indefinitely. If $x > 1$, we simply reverse the inequality and get the same conclusion. Therefore, we may write, $xd_p = d_p$ (where d_p is the principal element of d*), and since the elements of d* share this property we may write $xd^* = d^*$, meaning, that $xd = d$ for every element d of d*. We consider d* the equivalence class of its elements. In the case of $x + (0.1)^n$ and x, we look at the nth g-terms of each and, as n increases indefinitely, $x + (0.1)^n$ and x become indistinguishable. Now, since $(0,1)^n > ((0.1)^m)^n > 0$ and the extreme terms both tend to 0 as n increases indefinitely, so must the middle term tend to 0 so that they become indistinguishably small (the reason d* is called dark for being indistinguishable form 0 yet greater than 0): We summarize our discussion as follows:

if x is not a new integer, $x + d^* = x$; otherwise, if x = N.99… $x + d^* = N+1$, $x - d^* = x$; if $x \neq 0$, $xd^* = d^*$; $(d^*)^n = d^*$, n = 1, 2, …, N = 0, 1, … (12)

$1 - d^* = 0.99\ldots$, N – (N – 1).99… $1 - 0.99\ldots = d^*$, N = 1, 2, … (13)

It follows that the g-closure of **R**, i.e., its closure in the g-norm, is **R*** which includes the additive and multiplicative inverses and d*. We also include in **R*** the upper bounds of the divergent sequences of terminating decimals and integers (a sequence is divergent if the nth terms are unbounded as n increases indefinitely, e.g., the sequence 9, 99, …) called unbounded number u* which is countably infinite since the set of sequences is. We follow the same convention for u*: whenever we have a statement "u has property P for every element u of u*" we can simply say "u* has property P). Then u* satisfies these dual properties:

for all x, $x + u^* = u^*$; for $x \neq 0$, $xu^* = u^*$. (14)

Neither d* nor u* is a decimal and their properties are solely determined by their sequences. Then d* and u* have the following dual or reciprocal properties and relationship:

$0d^* = 0$, $0/d^* = 0$, $0u^* = 0$, $0/u^* = 0$, $1/d^* = u^*$, $1/u^* = d^*$. (15)

Numbers like $u^* - u^*$, d^*/d^* and u^*/u^* are still indeterminate but indeterminacy is avoided by computation with the g- or d-terms.

The decimals are linearly ordered by the lexicographic ordering "<" defined as follows: two elements of **R** are equal if corresponding digits are equal. Let

$N.a_1a_2..., M.b_1b_2... \in \mathbf{R}$. (16)

Then,

$N.a_1a_2... < M.b_1b_2$ if $N < M$ or if $N = M$, $a_1 < b_1$; if $a_1 = b_1$, $a< b_2$; …, (17)

and, if x is any decimal we have,

$0 < d^* < x < u^*$ (18)

The trichotomy axiom follows from lexicographic ordering. This is the natural ordering mathematicians sought among the real numbers but it does not exist there because it contradicts the trichotomy axiom.

5. DUALS AND THEIR RECIPROCALS

Mathematical systems are better understood by bringing in the notion of dual systems because it introduces some symmetry that may be useful. We can look at divergent sequences, i.e., sequences whose terms become bigger and bigger that we can no longer comprehend them and become indistinguishable from each other, as dual of convergent sequences. In this sense the divergent sequences also form a continuum. We denote their upper bounds by u* which satisfies (12) and (15). Then we look at d* as the dual of u* and **R*** that of the system of additive and multiplicative inverses (which has holes, namely, the nonexistent multiplicative inverses of integers). Thus **R*** is a semi-field, the nonzero integers forming a semi-ring since some of them have no multiplicative inverses. Like d*, u* cannot be separated from the decimals, i.e., there is no boundary between either of them and the decimals and between finite and infinite, i.e., we cannot separate d* from a decimals and there is no boundary to cross between finite and infinite so that beyond a certain finite decimal everything else is infinite. The latter is what is meant by the expression u* + x = u* for any decimal x. Duality is also seen in this case: Let $\lambda > 1$ be terminating decimal then the sequence λ^n, n = 1, 2, …, diverges to u* but $(1/\lambda)^n$, n = 1, 2, … converges, d-lim $(1/\lambda)^n$ = d*.

6. ISOMORPHISM BETWEEN THE INTEGERS AND DECIMAL INTEGERS

To find out more about the structure of **R*** we show the isomorphism between the integers and the decimal integers, i.e., integers of the form,

N.99…, N = 0, 1, … (19)

but before doing so we first note that 1 + 0.99… is not defined in **R** since 0.99… is nonterminating but we can write 0.99. . . = 1 – d* so that 1 + 0.99… = 1 + 1 – d* = 2 – d* =1.99… and we now define 1 + 0.99… = 1.99… or, in general, N – d* = (N−1).99… The pairs (N,(N–1).99…), N = 1, 2,..., are called twin integers because they are isomorphic:

Let f be the mapping N → (N – 1).99… then we show that f is an isomorphism between the integers and decimal integers.

(a) f(N+M) = (N+M–1).99… = N + M – 1 + 0.99…
= N – 1 + M – 1 + 1.99… = N – 1 + 0.99… + M – 1 + 0.99…
= (N–1).99… + (M–1).99… = f(N) + f(M). (20)

Equation (20) means that addition of integers is the same as addition of decimal integers.

Next, we show that multiplication is also an isomorphism.

(b) f(NM) = (NM–1).999... = NM – 1 + 0.99…

(c) = NM – N − M + 1 + N + –1 + M + –1 + 0.99…
= NM – N – M + 1 + (N–1).99… + (M–1).99… + (–1)(0.99…
= NM – N – M + 1 + N(0.99… + (–1)(0.99… + M(0.99…
+ (–1)(0.99…) + 0.99… = (N – 1)(M – 1) + (N–1)(0.99…)
+ (M – 1)(0.99… + $(0.99…)^2$ = ((N – 1) + 0.99… M – 1) + 0.99…
= (((N – 1).99…)((M – 1).99…) = (f(N))(f(M)). (21)

We have now established the isomorphism between the integers and the decimal integers with respect to both operations. We include in this isomorphism the map d* → 0, so that its kernel is the set {d*,0.99…) from which follows equations (22):

$(d^*)^n$ = d* and $(0.99…)^n$ = 0.99…, n = 1, 2, …. (22)

(The second equation can be proved also by mathematical induction)

For the curious reader we exhibit other properties of 0.99… Let K be an integer, M.99… and N.99… decimal integers. Then

(a) K + M.99… = (K+M).99…

(b) K(M.99…) = K(M + 0.99…) = KM + K(0.99…) = KM + (K–1).99…

(c) M.99… + N.99… = M + N + 0.99… + 0.99…). (23)

To verify that 2(0.999...) = 1.99…, we note that (1.99…)/2 = 0.99…

(d) (M.99…)(N.99…) = (M + 0.99…)(N + 0.99…)
= MN + M(0.99… + N(0.99…) + $(0.99…)^2$
= MN + (M–1).999… + (N–1).99… + 0.99…
= MN + (M + N–2).99… + 0.99…
= MN + (M + N–.1).99… = (MN+M+N–1).99…
(e) 0.99… + 0.99… = 2(0.99…) = 1.99… (24)

We extended the isomorphism to include d* by the mapping f(0) = d*, even if d* is neither a decimal nor an integer, because d* behaves like 0 and 0.99… like 1. The isomorphism makes the decimal integers also integers (i.e., equivalent and behave alike) in the sense of [3].

7. ADJACENT DECIMALS AND RECURRING 9s

Two decimals are adjacent if they differ by d*. Predecessor-successor pairs and twin integers are adjacent. In particular, 74.5700… and 74.5699… are adjacent.

Since the decimals have the form $N.a_1a_2…a_n,…$, N = 0, 1, 2, …, the digits are identifiable and, in fact, countably infinite and they are linearly ordered by lexicographic ordering. Therefore, they are discrete or digital and the adjacent pairs are also countably infinite. However, since their tail digits form a continuum, **R*** is a continuum with the decimals its countably infinite discrete subsystem.

A decimal is called recurring 9 if its tail decimal digits are all equal to 9. For example, 4.3299… and 299.99… are recurring 9s; so are the decimal integers. (In an isomorphism between two algebraic systems, their operations are interchangeable, i.e., they have the same algebraic structure but differ only in notation).

The recurring 9s have interesting properties. For instance, the difference between the integer N and the recurring 9, (N – 1).99…, is d*; such pair of decimals are called adjacent because there is no decimal between them and they differ by d*. In the lexicographic ordering the smaller of the pair of adjacent decimals is the predecessor and the larger the successor. The average between them is the predecessor. Thus, the average between 1 and 0.99… is 0.99… since (1.99…)/2 = 0.99…; this is true of any recurring 9, say, 34.5799… whose successor is 34.5800… Conversely, the g-limit of the iterated or successive averages between a fixed decimal and another decimal of the same integral part is the predecessor of the former.

Since adjacent decimals differ by d* and there is no decimal between them, i.e., we cannot split d* into nonempty disjoint sets, we have another proof that d* is a continuum (in the algebraic sense). Then we have another proof that **R*** is a continuum (also in the algebraic sense).

It follows from the counterexample to the trichotomy axiom that an irrational number cannot be expressed as limit of sequence of rationals since the closest it can get to it is some rational interval which still contains some rational whose relationship to it is unknown.

Now we know what the eurrationals are; they are the nonterminating decimals, periodic and nonperiodic. The g-sequence of an eurrational, which is a sequence of rationals, gets directly to its g-limit, digit by digit. We note further that an eurrational is an infinite series in terms of its digits as follows:

$$N.a_1a_2\ldots a_n\ldots, = N + .a_1 + .0a_2 + \ldots + .00\ldots0a_n + \ldots;\ 0.99\ldots = 0.9 + 0.09 + \ldots \quad (25)$$

8. THE STRUCTURE OF R* AND ITS SUBSPACES

We add the following results to the information we now have about the various subspaces of **R*** to provide a full picture of the structure of the new real number system. The next theorem is a definitive result about the continuum **R***.

Theorem. In the lexicographic ordering **R*** consists of adjacent predecessor-successor pairs (each joined by d*); therefore, the g-closure **R*** of **R** is a continuum [9].

Proof. For each N, N = 0, 1, …, consider the set of decimals with integral part N. Take any decimal in the set, say, $N.a_1a_2...$, and another decimal in it. Without loss of generality, let $N.a_1a_2...$ be fixed and let it be the larger decimal. We take the average of the nth g-terms of $N.a_1a_2...$and the second decimal; then take the average of the nth g-terms of this average and $N.a_1a_2...$; continue. We obtain the d-sequence with nth d-term, $(0.5)^{-n}a_1a_2...a_{n+k}$, which is a d-sequence of d*. Therefore, the g-limit of this sequence of averages is the predecessor of $N.a_1a_2...$ and we have proved that this g-limit and $N.a_1a_2...$ are predecessor-successor pair, differ by d* and form a continuum. Since the choice of $N.a_1a_2...$ is arbitrary then by taking the union of these predecessor-successor pairs of decimals in **R*** (each joined by the continuum d*) for all integral parts N, N = 0, 1, …, we establish that **R*** is a continuum. □

However, the decimals form countably infinite discrete subspace of R* since there is a scheme for labeling them by the integers.

We can imagine the terminating decimals as forming a right triangle with one edge horizontal and the vertical one extending without bounds. The integral parts are lined up on the vertical edge and they are joined together by their branching digits between the hypotenuse and the horizontal and extend to d* which is adjacent to 0 (i.e., differs from 0 by a dark number) at the vertex of the horizontal edge.

Corollary. **R*** is non-Archimedean and non-Hausdorff in both the standard and the g-norm and the subspace of decimals are countably infinite, hence, discrete but Archimedean and Hausdorff.

The following theorem is standard in the real in the real number system with the standard norm. Therefore, we do not bring in d* in the proof so that this is really a theorem about the decimals with the standard norm which is not true in the g-norm because the decimals merge into a continuum at their tail digits and cannot be separated.

Theorem. The rationals and irrationals are separated, i.e., they are not dense in their union (this is the first indication of discreteness of the decimals) [7].

Proof. Let $p \in \mathbf{R}$ (the real numbers including the ambiguous irrationals with the standard norm) be an irrational number and let q_n, n = 1, 2, …, be a sequence of rationals towards and left of p, i.e., n > m implies $q_n > q_m$; let d_n be the distance from q_n to p and take an open ball of radius $d_n/10^n$, center at q_n. Note that q_n tends to p but distinct from it for any n. Let $U = \cup U_n$, as $n \to \infty$, then U is open and if q is any real number, rational or irrational to the left of p then q is separated from p by disjoint open balls, one in U and, center at q and the other in the complement of U, center at p. Since the rationals are countable the union of set open sets U for all the rationals and the irrational p is separated from all the rationals.

We use the same argument if p were rational and since the *reals* has countable basis we take q_n an irrational number, for each n, at center of open ball of radius $d_n/10^n$. Take U to be the union of such open balls then, using the same argument, a real number in U, rational or irrational, is separated by disjoint open balls from p. □

This means that every decimal is separated from the rest, the terminating decimals from the eurrationals and from each other.

The next theorem has standard proof (in **R**); it raised eyebrows in internet forums.

Theorem. The largest and smallest elements of the open interval (0,1) are 0.99… and 1 – 0.99…, respectively [6].

Proof. Let C_n be the nth term of the g-sequence of 0.99… For each n, let I_n be open segment (segment that excludes its endpoints) of radius 10^{-2n} centered at C_n. Since C_n lies in I_n for each n, C_n lies in (0,1) as n increases indefinitely. Therefore, the decimal 0.99… lies in the open interval (0,1) and never reaches 1. To prove that 0.99… is the largest decimal in the open interval (0,1) let x be any point in (0,1). Then x is less than 1. Since C_n is steadily increasing n can be chosen large enough so that x is less than C_n and this is so for all subsequent values of n. Therefore, x is less than 0.99… and since x is any decimal in the open interval (0,1) then 0.99… is, indeed, the largest decimal in the interval and is itself less than 1.

To prove that 1 – 0.99… is the smallest element of **R**, we note that the g-sequence of 1 – 0.99… in (16) is steadily decreasing. Let K_n be the nth term of its g-sequence. For each n, let B_n be an open interval with radius 10^{-2n} centered at k_n. Then K_n lies in B_n for each n and all the B_ns lie in the open set in (0,1). If y is any point of (0,1), then y is greater than 0 and since the generating sequence 1 – 0.99… is steadily decreasing n can be chosen large enough such that y is greater than K_n and this is so for all subsequent values of n. Therefore, y is greater than = 1 – 0.99… and since the choice of y is arbitrary, 1 – 0.9… is the smallest number in the open interval (0,1); at the same time 1 – 0.99… is greater than 0. □

This theorem is true in the real number system and follows from the properties of the terminating decimals but it was not known because neither 0.99… nor 1 – 0.99… was well-defined; it was assumed all along that 1 = 0.99…the right side being ill-defined.

The next theorem used to be called Goldbach's conjecture [4,7].

Theorem. An even number greater than 2 is the sum of two prime numbers.

This is unsolved because, like Fermat's equation (FLT) [5], it is indeterminate. Before proving the theorem, we first note that an integer is a prime if it leaves a positive remainder when divided by another integer other than 1. We retain this definition in **R***.

Proof. The conjecture is obvious for small numbers. Let n be even greater than 10, p, q integers and p prime. If q is prime the theorem is proved; otherwise, it is divisible by an integer other than 1 and q. If we divide q by an integer other than 1 and q then since d* cannot be separated from any decimal, the remainder is d* > 0. Therefore, q is prime. □

We now have a sense of how the decimals are arranged by the lexicographic ordering. Consider the decimals with integral part N:

N.99……………

…………………

N.4800…………

N.4799…………

…………………

N.10……………..

…………………..

N.00…0100…

…………………..

N.00… (26)

The largest decimal in the set is the decimal integer N.99… and the smallest is the terminating decimal N.00… From the bottom up the decimals with integral part N are arranged as predecessor-successor pairs each joined by d*.Each gap indicated by the ellipses is filled by countably infinite adjacent predecessor-successor pairs each also joined by d* so that their union is a continuum. We now have a clear picture of how **R*** is arranged on the new real line linearly ordered by <, the lexicographic ordering.

A. Important results; resolution of a paradox

(1) Every convergent sequence has a g-subsequence that defines a decimal adjacent to its limit. If the decimal is terminating it is the limit itself.

(2) It follows from (1) that the limit of a sequence of terminating decimals can be found by evaluating the g-limit of its g-subsequence which is adjacent to it. We can use this as alternative way of computing the limit of ordinary sequence.

(3) In [10] several counterexamples to the generalized Jourdan curve theorem for n-sphere are shown where a continuous curve has points in both the interior and exterior of the n-sphere, n = 2, 3,. . . , without crossing the n-sphere. The explanation is: the functions cross the n-sphere through dark numbers.

(5) Given two decimals and their g-sequences and respective nth g-terms A_n, B_n we define the nth g-distance as the g-norm $|A_n - B_n|$ of the difference between their nth g-terms. Then their g-distance is the g-lim $|A_n - B_n|$, as $n \to \infty$, which is adjacent to the standard norm of the difference [3]. The advantage here is that the g-distance is the g-norm of their decimal difference and the difference between nonterminating decimals cannot be evaluated otherwise. Moreover, this notion of distance can be extended to n-space, n – 2, 3, .., and the distance between two points can be evaluated digit by digit in terms of their components without the need for evaluating roots. In fact, any computation in the g-norm yields the results directly, digit by digit, without the need for intermediate computation such as evaluation of roots in standard computation.

B. More on nonstandard numbers

We highlight some properties of special class of nonstandard numbers that can be checked by looking at their g- or divergent sequences. The principal element of d* (g-limit of its principal g-sequence) is dark number of order 0.1.

Let γ be a fraction such that $0 < \gamma < 1$ and let d_γ = g-limγ^n, as $n \to \infty$, n integer, d_γ is called dark number of order γ. An unbounded number u of order $\lambda > 1$ is defined as the upper bound of the sequence λ^n, as $n \to \infty$. The number u is an element of u* just as d_γ is an element of d*. Since γ^n is positive and steadily decreasing, d_γ is less than any given decimal. (In this section we only consider positive decimal and hence we shall drop the qualification positive) To see this, let x be any decimal; since $\gamma < 1$, the integer n can be chosen large enough that $0 < d_\gamma < \lambda^n < x$. Similarly, it can be an unbounded decimal of any order greater than any decimal.

The following is obvious by checking their g-sequences.

1. The product of any decimal and dark number of order γ is dark number of order γ; the product of a decimal and unbounded number of order λ is unbounded of order λ.

2. If d_1 and d_2 are dark numbers of order γ_1 and γ_2, respectively, where $\gamma_1 < \gamma_2$, then $d_1 + d_2$ is dark number of order γ_2, $d_2 - d_1$ is dark number of order γ_1, d_1/d_2 is dark number of order γ_1/γ_2 and d_1/d_2 is unbounded number of order γ_2/γ_1.

3. A decimal divided by a dark number of order γ is unbounded of order $1/\gamma$; a decimal divided by unbounded number of order λ is a dark number of order $1/\lambda$; the reciprocal of dark number of order γ is unbounded number of order $1/\gamma$; the reciprocal of unbounded number of order λ is dark number of order $1/\lambda$.

4. If μ_1, μ_2 are unbounded numbers of orders λ_1, λ_2, respectively, where $\lambda_1 > \lambda_2$, then $\mu_1 + \mu_2$ and $\mu_1 - \mu_2$ are both unbounded numbers of order λ_1 and μ_1/μ_2 and μ_2/μ_1 are unbounded and dark numbers of orders λ_1/λ_2 and λ_2/λ_1, respectively.

5. The sum of two dark numbers of the same order is a dark number of that order; the quotient of two dark numbers is indeterminate but can be avoided using nth g-term approximation. If the nth g-term of the quotient is a decimal then the quotient is a decimal, if it is greater than 1 the quotient is u*; if it is less than 1 the quotient is d*.

These results, taken from [6], are useful in avoiding indeterminate forms in calculation. Moreover, since all elements of d* share the properties of d*, we can use any element of this class for our argument in proving a theorem, especially, in dealing with inequality, the advantage being that it has clear structure. Consequently, there is no loss of generality in using the principal nth g-term of d* for any purpose involving d*.

Remark

Gauss' diagonal method proves neither the existence of nondenumerable set nor a continuum; it proves only the existence of countably infinite set, i.e., the off-diagonal elements consisting of countable union of countably infinite sets. The off-diagonal elements are not even well-defined because we know nothing about their digits (a decimal is determined by its digits). Therefore, we raise these conjectures:

Conjectures. (1) Nondenumerable set does not exist; (2) Only discrete set has cardinality; a continuum has none.

9. THE COUNTEREXAMPLES TO FLT

Given the contradiction in negative statement, we use Fermat's equation in place of the statement of Fermat's last theorem (FLT) so that its solutions are counterexamples to FLT. We first summarize the properties of the basic digit 9.

(1) String of 9s differs from nearest power of 10 by 1, e.g., $10^{100} - 99\ldots9 = 1$.

(2) If N is an integer, then $(0.99\ldots)^N = 0.99\ldots$ and, naturally, both sides of the equation have the same g-sequence. Therefore, for any integer N, $((0.99,..)10)^N = (9.99\ldots)10^N$.

(3) $(d^*)^N = d^*$; $((0.99,..)10)^N + d^* = 10^N$, N = 1, 2, …

Then the exact solutions of Fermat's equation are given by the triple (x,y,z) = $((0.99\ldots)10^T, d^*, 10^T)$, T = 1, 2, …, that clearly satisfies Fermat's equation,

$$x^n + y^n = z^n, \qquad (26)$$

for n = NT > 2. Moreover, for k = 1, 2, …, the triple (kx,ky,kz) also satisfies Fermat's equation. They are the countably infinite counterexamples to FLT that prove the conjecture false [5]. One counterexample is, of course, sufficient to disprove a conjecture.

10. WELL-BEHAVED FUNCTIONS

We call well-behaved polynomial, rational, exponential, logarithmic and circular functions and their sum, product, quotient and composites away from points of discontinuity. We consider well-behaved functions on the terminating decimals **R** which are discrete but continuous on **R*** being a continuum.

Since **R** is discrete its image under a well-behaved function is discrete. However, since the dark numbers between points in its graph are not seen the latter behaves like a continuous graph of standard analysis, the difference only of interest for computing and applications, especially, simulation where the tools are discrete. Then there is no need to approximate continuous function by discrete function as done in [14,15].

11. COMPUTATION

Computation is mapping of function or algebraic operation on functions into a number system or system of functions (e.g., finding solution of differential equation). Computation includes finding the result of algebraic operations on functions and evaluation of their values and limits.

We recall some of the concepts involved in computation. One is the limit point of topology (we simply refer to it as limit) in the standard norm which is clearly defined. The point P is limit of a sequence or series (sum of terms of a sequence) if every neighborhood of P contains a term of the sequence or series. What is its relationship with the g-norm? They are adjacent (differ by d*) because the standard norm of a decimal $N.a_1a_2\ldots a_n\ldots$ is the sum of its series expansion,

$$N.a_1a_2\ldots a_n\ldots = N + .a_1 + .0a_2 + \ldots + .0\ldots0\, a_n\ldots = N + \Sigma(1)^{-n}a_n,\ n = 1, 2, \ldots \qquad (27)$$

This sum in the standard norm is adjacent to the decimal; therefore, it approximates the decimal by an error of d*.

Since the g-norm is precise, i.e., yields the result of computation directly as a decimal digit by digit, the margin of precision is determined by the number of decimal digits computed; the intermediate steps of standard computation, e.g., evaluation of roots, are avoided. Then the limit is approximated by the g-limit, i.e., the decimal, to any level of accuracy within 10^{-n}, the nth d-term of d*. Thus, computation by the g-norm saves considerable computer time.

As example, we note that the standard norm or magnitude of the nonterminating decimal 0.99… is the sum of the series,

$$0.99\ldots = 0.9 + 0.09 + \ldots\ 0.00\ldots09 + \ldots = \Sigma9(1)^{-n}, \qquad (28)$$

which is and approximation of 1 at margin of error d* and 1 is adjacent to its g-norm, 0.99…, since 1 – 0.99… = d* [3]. Of course, d* does not show in (28) being dark but d* + 0.99… = 1.

For purposes of computation we denote the nth g-term of a decimal by the functional notation n-ξ(x) called n-truncation. Since a g-sequence defines or generates a decimal we call the latter its g-limit. Since nonterminating decimals cannot be added, subtracted, multiplied or divided, they must be n-truncated first to carry out the operations on them. The margin of error at each step in the computation must be consistent (analogous to the requirement of number of significant figures in physics, the rationale being that the result of computation cannot be more accurate than any of the approximations of the terms). While we can start division by terminating decimal since it starts on the left digits the quotient does not exist when the divisor has a prime factor other than 2 or 5. On this basis we have modified the definition of a rational as quotient of two integers except when the divisor has a prime other than 2 or 5. All the rest we simply call non-rationals. They include all the nonterminating decimals, periodic and nonperiodic.

Let $x = N.a_1 \ldots a_n \ldots$ and $y = M.b_1 \ldots b_n \ldots$, then

$$n\text{-}\xi(x) = N.a_1 \ldots a_n,\ n\text{-}\xi(y) = M.b_1 \ldots b_n,\ n\text{-}\xi(x + y) = n\text{-}\xi(x) + n\text{-}\xi(y), \quad (29)$$

$$n\text{-}\xi(x - y) = n\text{-}\xi(x) - n\text{-}\xi(y),\ n\text{-}\xi(xy) = (n\text{-}\xi(x))(n\text{-}\xi(y)),\ n\text{-}\xi(x/y) = (n\text{-}\xi(x))/(n\text{-}\xi(y)), \quad (30)$$

provided $n\text{-}\xi(y) \neq 0$ as divisor. Consider the function $f(x_1,\ldots,x_k)$ of several variables; we n-truncate f as follows:

$$n\text{-}\xi(f(x_1,\ldots,x_k)) = f(n\text{-}\xi(x_1),\ldots, n\text{-}\xi(x_k)). \quad (31)$$

If f is a composite function of several variables, $f(g_1(x_1,\ldots,x_t),\ldots,g_s(y_1,\ldots,y_u))$ then,

$$n\text{-}\xi(f(g_1(x_1,\ldots,x_t),\ldots,g_s(y_1,\ldots,y_u))) = f(n\text{-}\xi(g_1(n\text{-}\xi(x_1)),\ldots,n\text{-}\xi(x_t)),\ldots, n\text{-}\xi(g_s(n\text{-}\xi(y_1)),\ldots, n\text{-}\xi(y_u))). \quad (32)$$

This formalizes standard computation now based on the new real numbers. The computation itself uses the g-terms of the decimals involved and provides the result directly, digit by digit; it approximates the result to within any d-term of d*, the closest approximation one can ever get to is d* as in (28). Computation using the g-norm applies to monotone increasing function since the g-terms of a decimal is monotone increasing. However, a monotone decreasing function can be converted to a monotone increasing one and g-norm computation applied to the latter.

We give very simple examples below to illustrate the methodology without getting distracted by unnecessary complexity. Consider the monotone increasing function,

$$f(x) = x^{1/3}. \quad (33)$$

We want to evaluate f(5) to within 3 decimal digits. We make a series of 3-trunctions of f(5) to find the first three g-terms of its g-sequence.

Step 1. Find the largest integer N such that $N^3 \leq 5$. Clearly, N = 1.

Step 2. Divide segment [0,1] by points, 0, 0.1, 0.2, …, 0.9, 1, and find the largest number a_1 such that $(1.a_1)^3 \leq 5$. If $(1.a_1)^3 = 5$, f(5) is a terminating decimal. This is not so here since, $a_1 = 0.7$ and $(1.7)^3 = 4.913$ and the first term of the g-sequence is 1.7.

Step 3. Divide the segment [0,0.1] by the points 0, 0.01, 0.02, …, 0.09, 0.1] and find the largest number among them and call it a_2 such that $(1.7a_2)^3 \leq 5$. In this case, $a_2 = 0$. Then the first three digits of f(5) are known: 1.70.

Step 4. Find a_3 such that $(1.70a_3)^3 \leq 5$; then $a_3 = 9$ and $(1.709)^3 = 4.991$. Thus, the 3rd g-term of f(5) = 1.709. The calculation can be carried out to find the nth term of the g-sequence of f(5) for any n. This is how the scientific calculator computes cube root. Actually, we calculated the first three terms of the g-sequence of $x^{1/3}$ or its 3rd g-term.

This calculation applies to any well-behaved function since every point on it away from point of discontinuity has a neighborhood in which it is monotone. We used the g-norm to compute the result directly digit by digit. This is exactly how the calculator does it. It may look time-consuming but it can be done in split second with the right software.

Suppose we have the composite function, h(x) = f(g(x)), where $f(x) = x^{1/2}$, g(x) = x + 1 so that $f(g(x)) = (x + 1)^{1/2}$. We want to evaluate h(9) up to the 3rd decimal digit, i.e., at 10^{-3} margin of error.

Step 1. We want to find the 3-truncation $N.a_1a_2a_3$ of h(9). We first compute the integral part. Obviously, the largest integer N such that N^2 does not exceed $(h(9))^2 = 10$ is N = 3.

Step 2. Divide the interval [0,1] by the points, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1, and find the largest among them and denote it by $0.a_1$ such that $(3.a_1)^2$ does not exceed 10. that decimal is 0.1.

Step 3. Divide the interval [0,0.1] by the division points 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.1 and find the largest of the division points and denote it by $0.0a_2$ such that $(3.1a_2)^2$ does not exceed 10. The decimal is 0.06.

Step 4. Divide the interval [0,0.01] by the division points 0.001, 0.002, 0.003, 0.004, 0.005, 0.006, 0.007, 0.008, 0.009, 0.01 and find the largest among the division points and denote it by $0.00a_3$ such that $(3.16a_3)^2$ does not exceed 10. That number is 0.002.

Therefore, the 3rd g-term approximation of h(9) at margin of error $(10)^{-3}$ is 3.162.

To find fractional root of a decimal x, say, $h(x) = x^{k/m}$, we consider it as a composite function f(g(x)), where $f(x) = x^{1/m}$, $g(x) = x^k$ both of which are monotone increasing. Then we can n-truncate each of f(x) and g(x) and then n-truncate the corresponding composite of their n-trunctions to obtain the nth g-term approximation of the composite function h(x). Extension to sum, product and quotients is obvious.

12. THE LIMIT OF A DECIMAL

We recall that point P is the limit or limit point of the g-sequence or series expansion of a nonterminating decimal if every open interval containing P contains an element of its sequence or series. For example, 1 is the unique limit of 0.99… Since the limit and g-limit of a nonterminating decimal are adjacent the g-limit of the value of a function approximates the value at margin of error d* or the nth g-term at margin of error 10^{-n}. The terminating decimal 4.5300… and 4.5299… are adjacent the former being the successor of the latter in the lexicographic ordering of the decimals. Therefore, the former is the limit of the g-sequence of the latter. A nonstandard number, aside from d* and u*, is the sum of a decimal and d*. This means that its g-sequence has a set of digits that moves to the right indefinitely leaving fixed digits behind. The digits that move to the right are the nonstandard nth d-terms of d* and the digits that remain fixed are the digits of the g-terms of the decimal. Since d* cannot be separated from the decimal as its dark component except that we cannot identify its dark component being a point in the continuum we can look at the latter as the standard component. Moreover, d* cannot be separated from 0 although distinct from it; therefore, it is adjacent to it.

The sequence, 1.25315, 1.250153, 1.2500351, 1.25000531, … shows the nth d-terms of some nonstandard decimal, i.e., a terminating decimal, and the receding d-terms of d* so that the nonstandard decimal is 1.25 + d* that reduces to the standard decimal 1.25. In other words, we can look at a decimal as approximation of some nonstandard number and the margin of error is d*. What is the purpose of all these? Consider the function,

$$F(x) = H(x) + \delta(x), \quad (34)$$

where H(x) does not diverge as x tends to some limit and δ(x) tends to 0 as limit then in the calculation of the limit of F(x) as $x \to s$ the nth g-term will consist of two components one with digits remaining fixed and another set of digits that recedes indefinitely to the right, the nth d-term of d*. This means that in evaluating limit of a function we keep computing the value of the function (iterated computation) over finer and finer refinements of the sequence of termininating decimals that tends to s as limit (there is no loss of generality in taking successive averages between the terms of the sequence and s). We call H(x) and δ(x) the principal and minor parts of F(x), respectively. Even if the function is not separated into principal and minor parts, they will show in the calculation of its limit. Moreover, since the computation involves iterated approximations the problem of indeterminacy does not arise.

If the function is a sequence of terminating decimals $\{a_n\}$, n = 1, 2, …, we take the values of the sequence along n, which need not be consecutive values. To facilitate convergence we may skip some values of n and take large values.

Consider our previous example, the sequence of numbers, $f(n) = (n^4 + 1)/n^4$. We compute the terms along n = 1, 2, …, and note their truncated sequence of values:

n = 1, f(1) = 2.0000000
n = 2, f(2) = 1.0625000
n = 3, f(3) = 1.0123456
n = 4, f(4) = 1.0039062
…………………………

n = 50, f(50) = 1.0000000. (35)

Discarding the first term f(1) = 2.0000000 corresponding to n = 1, which is not a g-term, we have the nonstandard g-sequence of f(n),

1.0625000, 1.0625000, 1.0123456, 1.0039062, 1.0016000, …, 1.0001000, …, (36)

whose limit is 1 + d*. Note the nth d-term with set of digits varying and receding to the right indefinitely leaving the fixed digits behind. The varying elements are the d-terms of d*. The fixed digits left behind are the g-terms of 1, a terminating decimal.

Consider the limit of the sequence, $f(n) = (2n^4+1)/3n^4$, n = 1, 2, … We find a nonterminating decimal that is adjacent to it. We do the following computation:

n = 2,	f(2) = 0.6875000	n = 6,	f(6)	=	0.66669238
n = 4,	f(4) = 0.6679687	n = 7,	f(7)	=	0.66680540
n = 3,	f(3) = 0.6707818	n = 8,	f(8)	=	0.66674800
n = 5,	f(5) = 0.6672000	n = 9,	f(9)	=	0.66671740

……………………………………………………………………..

n = 100,   f(100) = 0.66666666.   (37)

Thus, the nonterminating decimal adjacent to limit of f(n), as $n \to \infty$, is 0.66…, a periodic.

We now compute the g-limit of a function. Let f(x) be a function and suppose we want to find limf(x), as $x \to s$. We find limf(x) along successive refinements starting with the steadily increasing sequence $x_0, x_1, x_2, \ldots$, left of and towards s > 0 as limit. We refine the sequence by inserting the succession of averages of s and x_0, of s and x_1, etc., …, relabeling them as $s_1, s_2, s_3, \ldots$, etc. so that the refinement becomes the sequence, $x_0, s_1, s_2, s_3, \ldots$ We continue the refinement and compute the values of the nth g-term of g-limf(x) along the kth refinement. If for some suitable value of k we find some set of digits in the nth g-terms of f(x) that recedes to the right leaving fixed digits behind up to the nth term, then that would be the nth g-term of the g-limit which is adjacent to the limit. If s is nonterminating, we obviously need to truncate s to the desired accuracy to do the computation. The advantages of this scheme for computing the value of f(x) or the limf(x) is quite clear. For instance, a software for computing limit of well-behaved functions can be developed that would take split second to get the result. This is not possible for non-well-behaved functions such as wild oscillation [8].

13. COMPUTATION WITH NONSTANDARD NUMBERS

Consider the function f(x) = g(x) + d(x) in the neighborhood of a decimal s, where x is decimal and g(x) (principal part) and d(x) (minor part) are decimal-valued functions and g(x) tends to nonzero decimal as limit and d(x) tends to 0, as $x \to s$. Then g-limf(x) = g-lim(g(x) + g-limd(x)) = g-limg(x), as $x \to a$. If g-limg(x) is unbounded at s then g-limf(x) is unbounded.

In computation we treat nonstandard function f(x) as binomial, the sum of principal and minor parts. In algebraic operations involving sum and product we write the result in the form $F(x) = G(x) + \eta(x)$, where F(x) and $\eta(x)$ are the principal and minor parts, respectively. Then g-limF(x) = g-limG(x), as $x \to s$. The minor part of the function G(x) may be discarded in the calculation of its g-limit; if they appear as factors, the arithmetic of dark numbers applies.

If λ is a terminating decimal, i.e., a fraction, such that $0 < \lambda < 1$, the g-limit of the nonstandard sequence λ^n, n = 1, 2, …, is called dark number of order λ. Consider the non-uniformly bounded convergent sequence S [6]:

$0.123, (0.312)^2, (0.231)^2, (0.123)^3,\ldots,$ (37)

whose terms are cyclic permutations of the digits 1, 2, 3. To find its limits we split it into three component sequences:

(a) $0.123, (0.123), (0.123)^3, \ldots,$
(b) $0.312), (0.312)^2,$
$(0.312)^3,\ldots,$ (c) $0.231, (0.231)^2, (0.231)^3,\ldots,$ (38)

which converge to distinct elements of d*. Therefore, the g-limit of (38) is three-valued, consisting of dark numbers of these orders. Since the number of ways of forming such component sequences is countable one may form a dark number of countable set-valued order as well as nonstandard functions with set-valued principal and minor parts. Since they all recede to the right indefinitely they become indistinguishable and their d limit is a continuum.

A module $H(x_1,. . .,x_k)$ is a rational expression in the variables $x_1, . . ., x_k$. Suppose the values of the arguments are given and H is computable as a single decimal, terminating or nonterminating. Then the value of H can be computed to any margin of error. If this is not the case, then compute the nth g-terms of the arguments, at consistent margin of error, and find the value of H in terms of those nth g-terms. Given two modules H and G and using the approximation function ξ_n, we define $\xi_n(H + G) = \xi_n(H) + \xi_n(G)$ and $\xi_n(HG) = \xi_n(H)_n\xi_n(G)$ at consistent margin of error. In dealing with nonterminating decimals it may not be possible to verify equality between two modules by actual computation. In this case, we say that two modules $H(x_1,\ldots,x_k)$, $G(x_1,\ldots,x_m)$ are equal if their n-truncations are equal, i.e., $\xi_n(H) = \xi_n(G)$, for each n = 1, 2, …

Although dark numbers of some orders are special elements of d* they share the properties of d* and we may substitute d* in any equation or expression involving them.

Consider the function H(x) = h(x) + d(x), where h(x) tends to a decimal and d(x) tends to 0, as $x \to s$. Then g-limH(x) = g-limh(x) since the sum of a decimal and dark number is the same decimal. If some function $\nabla g(x)$ is small and tends to 0 as limit, i.e., dark number of some order γ, and limh(x) is a decimal then $\lim(\nabla g(x)H(x)$, $x \to s$, is dark number of order γ provided h(x) does not diverge. A function of the form $\nabla g(x)hx)$, as $x \to s$, is dark number of order α if $\nabla g(x)$ tends to the form α^n, where $n \to \infty$, as $x \to s$. Note that g-limH(x) = g-limh(x) + g-limd(x) = g-limh(x). Thus, taking the g-limit of a function amounts to discarding the minor part of the nonstandard function provided the principal part does not diverge. If the g-limit is terminating decimal it is generally obtained by substitution because it is actually attained. For example, if f(x) is the principal part of some nonstandard function, say, H(x) = x^2 + d(x), then g-limH(x), as $x \to a$, is a^2. The sum or product of nonstandard functions is obtained by considering each function as binomial, the sum of its principal and minor parts. This is also the way to handle nonstandard numbers and operations and inverse operations. If the divisor tends to 0 then it is a divergent sequence of u* and the arithmetic of u* applies. For instance, if the quotient has the form $d_1(x)/d_2(x)$ and numerator and denominator tend to 0, i.e., dark numbers of orders γ_1, γ_2, respectively, the quotient has order $\gamma = \gamma_1/\gamma_2$ and is either a dark number, decimal or unbounded number depending on whether $\gamma < 1$, $\gamma = 1$ or $\gamma > 1$. However, by n-truncation to find the nth g-terms, we can tell what the g-limit would be and so indeterminacy is avoided. Moreover, for finding limit of ordinary indeterminate form, truncation will compute the g-limit directly without being bothered by indeterminacy.

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