http://users.tpg.com.au/nanahcub/pmvam.pdf

Professional Mathematics Versus Amateur Mathematics

By Derek Buchanan

Professional mathematics and amateur mathematics seem to coexist in many libraries and on the internet. This, together with disparate views on what to teach and how to teach it, are cause for much consternation and angst between professional mathematicians and amateur communicators of mathematics which warrants comment. Let's distinguish these two groups respectively with the names *the mathematician* and *Mickey Mouse*.

I should start by making it clear that I am definitely on the side of the mathematician - so this document will deliberately be a one-sided view.

I will discuss three examples which illustrate the problem - induction, the division algorithm and rational linear transformations. Many others exist. I may update this file to include some more in the future.

Induction mantras.

Both Mickey Mouse and the mathematician would do the basis step and induction step.

But Mickey Mouse would end it with some version of the following: *The statement is true for* $n = 0$ *and hence is true for* $n = 1$*. The statement is true for* $n = 1$ *and hence is true for* $n = 2$*. The statement is true for* $n = 2$ *and hence is true for* $n = 3$ *and so on. Hence the statement is true for all integers* $n \geq 0$ *by induction.* He seems to like wasting much time writing such a lengthy final statement.

The mathematician on the other hand would either not make any such mantra at all and merely put an end-of-proof symbol, $\square$, or he or she might make a better mantra with a simple statement like: *Hence the statement is true for all* $n \geq 0$ *by the principle of mathematical induction.*

Existence and uniqueness of the division transformation for polynomials

Mickey Mouse is quite comfortable with teaching that a polynomial f can be divided by another polynomial g with $\deg g \leq \deg f$ and

$$f = gq + r$$

for polynomials q and r with $\deg r < \deg g$ without proving the existence and uniqueness of q and r. Mickey Mouse will then tell students to go and use it to do lots of examples of long division exercises.

The mathematician has no such wonderful confidence, is not at all comfortable with it and will go and prove existence and uniqueness of q and r before telling students to use it to do long divisions.

Theorem. Let f, $g \in \mathbb{R}[X]$ be polynomials with $g \neq 0$. Then there exist unique polynomials q, $r \in \mathbb{R}[X]$ such that $f = qg + r$ and $\deg r < \deg g$.

Proof.

EXISTENCE OF q, r.

If $\deg g > \deg f$ then we set $q = 0$ and $r = f$. Otherwise, $\deg f \geq \deg g$. Let $f = \sum_{j=0}^{m} a_j X^j$ where $a_m \neq 0$ and $g = \sum_{j=0}^{n} b_j X^j$ where $b_n \neq 0$ and define $d = m - n \geq 0$. We use induction on d.

Base Step. Let $d = 0$, then $m = n$. We set $q = a_m/b_m$ and $r = f - qg$. Notice that q is well-defined because $b_m \neq 0$ and that the coefficient of X^m in r vanishes, whence $\deg r < m = \deg g$.

Induction Step. Now we assume that this is true whenever $d < k$ and consider $d = k$, so that $m = n + k$. Let $f_1 = f - (a_m/b_n)X^{m-n}g$. Notice that $\deg f_1 < \deg f$, whence by the induction hypothesis there exist q_1, r with $\deg r < \deg g$ and $f_1 = q_1 g + r$. Therefore $f = f_1 + \frac{a_m}{b_n}X^{m-n}g = (q_1 + \frac{a_m}{b_n}X^{m-n})g + r$, whence define $q = q_1 + (a_m/b_n)X^{m-n}$ and the result is true for $d = k$.

UNIQUENESS OF q, r.

Suppose that $f = qg + r = Qg + R$, with $\deg r < \deg g$ and $\deg R < \deg g$. Rearranging, we have $R - r = (q - Q)g$, whence g divides $R - r$. Since $\deg(R - r) < \deg g$, this can only happen if $R - r = 0$, whence $R = r$. In this case, $g(q - Q) = 0$, which since $g \neq 0$ implies that $q - Q = 0$ or, equivalently, that $Q = q$ $\square$

Rational linear transformations

Problem. Prove that if the locus of z is a circle, $\mathcal{C}$, not passing through the origin, then the locus of $\frac{1}{z}$ is a circle.

Mickey Mouse will insist we find the centre and radius. The mathematician does not.

Mickey Mouse's Solution. Let $\mathcal{C}_1 : |z - a| = r$ be the locus of z not through 0.

$a = \alpha + i\beta$

$z = x + iy$

$x^2 + y^2 - 2\alpha x - 2\beta y + k = 0$ for a constant k

$(x - \alpha)^2 + (y - \beta)^2 = \alpha^2 + \beta^2 - k = r^2$

$k = \alpha^2 + \beta^2 - r^2 = |a|^2 - r^2 \neq 0$ since $\mathcal{C}_1$ does not pass through 0.

$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{x}{|z|^2} - i\frac{y}{|z|^2}$

$|z|^2 - 2\alpha\Re(z) - 2\beta\Im(z) + k = 0$

Replace z with $\frac{1}{z}$

$|\frac{1}{z}|^2 - 2\alpha\Re(\frac{1}{z}) - 2\beta\Im(\frac{1}{z}) + k = 0$

$\frac{1}{k|z|^2} - \frac{2\alpha x}{k|z|^2} + \frac{2\beta y}{k|z|^2} + 1 = 0$

$|z|^2 - \frac{2\alpha x}{k} + \frac{2\beta y}{k} + \frac{1}{k} = 0$

$\mathcal{C}_2 : (x - \frac{\alpha}{k})^2 + (y + \frac{\beta}{k})^2 = \frac{\alpha^2+\beta^2}{k^2} - \frac{1}{k} = \frac{\alpha^2+\beta^2-k}{k^2} = \frac{r^2}{k^2}$

which is a circle, centre $\frac{\alpha}{k} - \frac{\beta}{k}i$ and radius $\frac{r}{|k|}$

i.e., the locus of $\frac{1}{z}$ is a circle, $\mathcal{C}_2$, with centre $\frac{\bar{a}}{|a|^2-r^2}$, radius $\frac{r}{||a|^2-r^2|}$.

Mathematician's Solution. Suppose z_1, z_2, z_3 lie on $\mathcal{C}$. Then $\mathcal{C}$ is

$$\begin{vmatrix} |z|^2 & \Re(z) & \Im(z) & 1 \\ |z_1|^2 & \Re(z_1) & \Im(z_1) & 1 \\ |z_2|^2 & \Re(z_2) & \Im(z_2) & 1 \\ |z_3|^2 & \Re(z_3) & \Im(z_3) & 1 \end{vmatrix} = 0$$

Dividing through by $|zz_1z_2z_3|^2$ (which we can do since it can't be 0)

$$\begin{vmatrix} 1 & \Re(\frac{1}{z}) & \Im(\frac{1}{z}) & |\frac{1}{z}|^2 \\ 1 & \Re(\frac{1}{z_1}) & \Im(\frac{1}{z_1}) & |\frac{1}{z_1}|^2 \\ 1 & \Re(\frac{1}{z_2}) & \Im(\frac{1}{z_2}) & |\frac{1}{z_2}|^2 \\ 1 & \Re(\frac{1}{z_3}) & \Im(\frac{1}{z_3}) & |\frac{1}{z_3}|^2 \end{vmatrix} = 0$$

Hence $\frac{1}{z}$ lies on a circle (in particular the circle through $\frac{1}{z_1}$, $\frac{1}{z_2}$, $\frac{1}{z_3}$).