

Pi Squared On Six

$$\mathbf{2002\ Ext.\ 2\ hsc\ Q8a} \Rightarrow \sum_{j=1}^m \cot^2 \frac{\pi j}{2m+1} = \frac{m(2m-1)}{3}$$

$$\therefore \sum_{j=1}^m (\operatorname{cosec}^2 \frac{\pi j}{2m+1} - 1) = \left(\sum_{j=1}^m \operatorname{cosec}^2 \frac{\pi j}{2m+1} \right) - m = \frac{m(2m-1)}{3}$$

$$\Rightarrow \sum_{j=1}^m \operatorname{cosec}^2 \frac{\pi j}{2m+1} = \frac{m(2m-1)}{3} + m = \frac{2m(m+1)}{3}.$$

For $0 < \theta < \frac{\pi}{2}$, $\sin \theta < \theta < \tan \theta$. $\therefore \cot \theta < \frac{1}{\theta} < \operatorname{cosec} \theta$ $\therefore \cot^2 \theta < \frac{1}{\theta^2} < \operatorname{cosec}^2 \theta$.

$$\therefore \frac{m(2m-1)}{3} = \sum_{j=1}^m \cot^2 \frac{\pi j}{2m+1} < \sum_{j=1}^m \frac{(2m+1)^2}{(\pi j)^2} = \frac{(2m+1)^2}{\pi^2} \sum_{j=1}^m \frac{1}{j^2} < \sum_{j=1}^m \operatorname{cosec}^2 \frac{\pi j}{2m+1} = \frac{2m(m+1)}{3}$$

$$\therefore 1 - \frac{4m(m+1)}{(2m+1)^2} = \frac{1}{(2m+1)^2} < 1 - \frac{6}{\pi^2} \sum_{j=1}^m \frac{1}{j^2} < 1 - \frac{2m(2m-1)}{(2m+1)^2} = \frac{6m+1}{(2m+1)^2} < \frac{6m+3}{(2m+1)^2} = \frac{3}{2m+1}$$

$$\therefore 0 = \lim_{m \rightarrow \infty} \frac{1}{(2m+1)^2} \leq \lim_{m \rightarrow \infty} \left(1 - \frac{6}{\pi^2} \sum_{j=1}^m \frac{1}{j^2} \right) = 1 - \frac{6}{\pi^2} \sum_{j=1}^{\infty} \frac{1}{j^2} \leq \lim_{m \rightarrow \infty} \frac{3}{2m+1} = 0$$

$$\text{Hence } \sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}.$$

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