

International Mathematical Olympiad 2007

1. Real numbers a_1, a_2, \dots, a_n are given. For each i , ($1 \leq i \leq n$), define

$$d_i = \max\{a_j | 1 \leq j \leq i\} - \min\{a_j | i \leq j \leq n\}$$

and let $d = \max\{d_i | 1 \leq i \leq n\}$.

- (a) Prove that, for any real numbers $x_1 \leq x_2 \leq \dots \leq x_n$,

$$\max\{|x_i - a_i| | 1 \leq i \leq n\} \geq \frac{d}{2}. \quad (*)$$

- (b) Show that there are real numbers $x_1 \leq x_2 \leq \dots \leq x_n$ such that the equality holds in (*).

2. Consider five points A, B, C, D and E such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let ℓ be a line passing through A . Suppose that ℓ intersects the interior of the segment DC at F and intersects line BC at G . Suppose also that $EF = EG = EC$. Prove that ℓ is the bisector of angle DAB .
3. In a mathematical competition some competitors are friends. Friendship is always mutual. Call a group of competitors a *clique* if each two of them are friends. (In particular, any group of fewer than two competitors is a clique.) The number of members of a clique is called its *size*.

Given that, in this competition, the largest size of a clique is even, prove that the competitors can be arranged into two rooms such that the largest size of a clique contained in one room is the same as the largest size of a clique contained in the other room.

4. In triangle ABC the bisector of angle BCA intersects the circumcircle again at R , the perpendicular bisector of BC at P , and the perpendicular bisector of AC at Q . The midpoint of BC is K and the midpoint of AC is L . Prove that the triangles RPK and RQL have the same area.
5. Let a and b be positive integers. Show that if $4ab - 1$ divides $(4a^2 - 1)^2$, then $a = b$.
6. Let n be a positive integer. Consider

$$S = \{(x, y, z) | x, y, z \in \{0, 1, \dots, n\}, x + y + z > 0\}$$

as a set of $(n + 1)^3 - 1$ points in three-dimensional space. Determine the smallest possible number of planes, the union of which contains S but does not include $(0, 0, 0)$.