Irrationality of e aka 1993 4 unit paper Q7b

(For $n \in \mathbb{Z}^+$, let $s_n = \sum_{r=0}^{n} \frac{1}{r!}$. Prove by induction that $e - s_n = e\int_0^1 \frac{x^n}{n!}e^{-x}\,dx$ and deduce that $0 < e - s_n < \frac{3}{(n+1)!}$, $(e-s_n)n! \notin \mathbb{Z}$ and $e \notin \mathbb{Q}$.)

$$\begin{aligned}
e\int_0^1 \tfrac{x^1}{1!}e^{-x}\,dx &= e\int_0^1 x\tfrac{d}{dx}(-e^{-x})\,dx \\
&= e([-xe^{-x}]_0^1 - \int_0^1 -e^{-x}\tfrac{dx}{dx}\,dx) \\
&= e(-e^{-1} - [e^{-x}]_0^1) \\
&= -1 - e(e^{-1} - 1) \\
&= e - 2 \\
&= e - \textstyle\sum_{r=0}^{1} \tfrac{1}{r!} \\
&= e - s_1
\end{aligned}$$

If $e - s_k = e\int_0^1 \frac{x^k}{k!}e^{-x}\,dx$ for some $k \in \mathbb{Z}^+$, then

$$\begin{aligned}
e - s_{k+1} &= (e - s_k) - \tfrac{1}{(k+1)!} \\
&= e\int_0^1 \tfrac{x^k}{k!}e^{-x}\,dx - \tfrac{1}{(k+1)!} \\
&= e\int_0^1 \tfrac{e^{-x}}{k!}\tfrac{d}{dx}\left(\tfrac{x^{k+1}}{k+1}\right)dx - \tfrac{1}{(k+1)!} \\
&= e(\left[\tfrac{e^{-x}}{k!}\cdot\tfrac{x^{k+1}}{k+1}\right]_0^1 - \int_0^1 \tfrac{x^{k+1}}{k+1}\cdot\tfrac{d}{dx}\left(\tfrac{e^{-x}}{k!}\right)dx) - \tfrac{1}{(k+1)!} \\
&= e(\tfrac{e^{-1}}{(k+1)!} + \int_0^1 \tfrac{x^{k+1}}{(k+1)!}e^{-x}\,dx) - \tfrac{1}{(k+1)!} \\
&= e\int_0^1 \tfrac{x^{k+1}}{(k+1)!}e^{-x}\,dx
\end{aligned}$$

and since $e - s_n = e\int_0^1 \frac{x^n}{n!}e^{-x}\,dx$ is true for $n = 1$, then by induction

$$e - s_n = e\int_0^1 \tfrac{x^n}{n!}e^{-x}\,dx \text{ for all } n \in \mathbb{Z}^+ \quad \ldots\ldots\ldots\ldots\ldots\ldots (\dagger)$$

For $0 < x < 1$, $0 < \frac{x^n}{n!} < \frac{1}{n!}$ and $\frac{1}{e} < e^{-x} < 1$ and hence $0 < \frac{x^n}{n!}e^{-x} < \frac{x^n}{n!}$ and therefore $0 < e\int_0^1 \frac{x^n}{n!}e^{-x}\,dx \le e\int_0^1 \frac{x^n}{n!}\,dx = \frac{e}{n!}\left[\frac{x^{n+1}}{n+1}\right]_0^1 = \frac{e}{(n+1)!} < \frac{3}{(n+1)!}$ since $e < 3$ and hence from $(\dagger)$, for $n \in \mathbb{Z}^+$, $0 < e - s_n < \frac{3}{(n+1)!}$ and therefore $0 < (e - s_n)n! < \frac{3}{(n+1)} \le 1$ for $n \in \mathbb{Z}^+\backslash\{1\}$ and moreover, since $2 < e < 3$, $0 < e - 2 = (e-2)1! = (e - s_1)1! < 1$ and so for any $n \in \mathbb{Z}^+$, $0 < (e - s_n)n! < 1$ and hence

$$\text{for any } n \in \mathbb{Z}^+,\ (e - s_n)n! \notin \mathbb{Z} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (*)$$

If $e \in \mathbb{Q}$ and $n \ge q$, then there exist $p, q \in \mathbb{Z}$ such that $e = \frac{p}{q}$ and $\therefore (e - s_n)n! = \frac{pn!}{q} - \sum_{r=0}^{n} \frac{n!}{r!} \in \mathbb{Z} \because r!|n!$ and $q|n!$, contradicting $(*)$. Hence $e \notin \mathbb{Q}$. $\square$