

Irrationality of e aka 1993 4 unit paper Q7b

(For $n \in \mathbb{Z}^+$, let $s_n = \sum_{r=0}^n \frac{1}{r!}$. Prove by induction that $e - s_n = e \int_0^1 \frac{x^n}{n!} e^{-x} dx$ and deduce that $0 < e - s_n < \frac{3}{(n+1)!}$, $(e - s_n)n! \notin \mathbb{Z}$ and $e \notin \mathbb{Q}$.)

$$\begin{aligned} e \int_0^1 \frac{x^1}{1!} e^{-x} dx &= e \int_0^1 x \frac{d}{dx} (-e^{-x}) dx \\ &= e([-xe^{-x}]_0^1 - \int_0^1 -e^{-x} \frac{dx}{dx} dx) \\ &= e(-e^{-1} - [e^{-x}]_0^1) \\ &= -1 - e(e^{-1} - 1) \\ &= e - 2 \\ &= e - \sum_{r=0}^1 \frac{1}{r!} \\ &= e - s_1 \end{aligned}$$

If $e - s_k = e \int_0^1 \frac{x^k}{k!} e^{-x} dx$ for some $k \in \mathbb{Z}^+$, then

$$\begin{aligned} e - s_{k+1} &= (e - s_k) - \frac{1}{(k+1)!} \\ &= e \int_0^1 \frac{x^k}{k!} e^{-x} dx - \frac{1}{(k+1)!} \\ &= e \int_0^1 \frac{e^{-x}}{k!} \frac{d}{dx} \left(\frac{x^{k+1}}{k+1} \right) dx - \frac{1}{(k+1)!} \\ &= e \left(\left[\frac{e^{-x}}{k!} \cdot \frac{x^{k+1}}{k+1} \right]_0^1 - \int_0^1 \frac{x^{k+1}}{k+1} \cdot \frac{d}{dx} \left(\frac{e^{-x}}{k!} \right) dx \right) - \frac{1}{(k+1)!} \\ &= e \left(\frac{e^{-1}}{(k+1)!} + \int_0^1 \frac{x^{k+1}}{(k+1)!} e^{-x} dx \right) - \frac{1}{(k+1)!} \\ &= e \int_0^1 \frac{x^{k+1}}{(k+1)!} e^{-x} dx \end{aligned}$$

and since $e - s_n = e \int_0^1 \frac{x^n}{n!} e^{-x} dx$ is true for $n = 1$, then by induction

$$e - s_n = e \int_0^1 \frac{x^n}{n!} e^{-x} dx \text{ for all } n \in \mathbb{Z}^+ \dots\dots\dots (\dagger)$$

For $0 < x < 1$, $0 < \frac{x^n}{n!} < \frac{1}{n!}$ and $\frac{1}{e} < e^{-x} < 1$ and hence $0 < \frac{x^n}{n!} e^{-x} < \frac{x^n}{n!}$ and therefore $0 < e \int_0^1 \frac{x^n}{n!} e^{-x} dx \leq e \int_0^1 \frac{x^n}{n!} dx = \frac{e}{n!} \left[\frac{x^{n+1}}{n+1} \right]_0^1 = \frac{e}{(n+1)!} < \frac{3}{(n+1)!}$ since $e < 3$ and hence from (\dagger) , for $n \in \mathbb{Z}^+$, $0 < e - s_n < \frac{3}{(n+1)!}$ and therefore $0 < (e - s_n)n! < \frac{3}{(n+1)} \leq 1$ for $n \in \mathbb{Z}^+ \setminus \{1\}$ and moreover, since $2 < e < 3$, $0 < e - 2 = (e - 2)1! = (e - s_1)1! < 1$ and so for any $n \in \mathbb{Z}^+$, $0 < (e - s_n)n! < 1$ and hence

$$\text{for any } n \in \mathbb{Z}^+, (e - s_n)n! \notin \mathbb{Z} \dots\dots\dots (*)$$

If $e \in \mathbb{Q}$ and $n \geq q$, then there exist $p, q \in \mathbb{Z}$ such that $e = \frac{p}{q}$ and $\therefore (e - s_n)n! = \frac{pn!}{q} - \sum_{r=0}^n \frac{n!}{r!} \in \mathbb{Z} \because r!|n!$ and $q|n!$, contradicting $(*)$. Hence $e \notin \mathbb{Q}$. \square