HOW NOT TO FIND THE SURFACE AREA OF REVOLUTION

By Derek Buchanan

Obviously one wouldn't follow the grotesque method suggested in the 2008 HSC exam.

First, I'll show how it should be done, and then how it shouldn't.

Method 1. The surface area of a sphere is simply $\int_{-r}^{r} 2\pi y\sqrt{1+(y')^2}\,dx$ where $y=\sqrt{r^2-x^2}$, whereupon we get

$$\begin{aligned}
\text{S.A.} &= \int_{-r}^{r} 2\pi\sqrt{r^2-x^2}\sqrt{1+\big(\frac{d}{dx}\sqrt{r^2-x^2}\big)^2}\,dx \\
&= \int_{-r}^{r} 2\pi r\,dx \\
&= 2\pi r[x]_{-r}^{r} \\
&= 2\pi r(r--r) \\
&= 4\pi r^2
\end{aligned}$$

That's nice. And that's how it should be done.

Now we proceed with a very bad method indicating that the Board of Studies is trying to train students with bad mathematics and that if teachers want to teach mathematics properly, they should ignore the Board of Studies and seek guidance elsewhere.

Method 2. The points P_0, P_1, ..., P_n are equally spaced in the first quadrant on the circular arc of radius R and centre O. The point P_0 is $(R,0)$, P_n is $(0,R)$ and $\angle P_{k-1}OP_k=\delta$ for $k=1$, ..., n. Each of the intervals $P_{k-1}P_k$ is rotated about the y-axis to form S_k, a frustrum of a cone.

If x_k and x_{k-1} are the x-coordinates of P_k and P_{k-1} then $x_k = R\cos k\delta$ and $x_{k-1}=R\cos(k-1)\delta$.

The nett of a frustrum of a cone is a sector of an annulus, and if the arcs of the sector are ℓ_k, ℓ_{k-1}, and the distance between them, d, the area is $\frac{1}{2}(\ell_k+\ell_{k-1})d$.

Now we have by the cosine rule $d=P_{k-1}P_k=\sqrt{R^2+R^2-2R\cdot R\cos\delta}=2R\sin\frac{\delta}{2}$ and $\ell_k=2\pi x_k$, $\ell_{k-1}=2\pi x_{k-1}$ and so if the area of S_k is A_k, we have

$$
\begin{aligned}
A_k &= \tfrac{1}{2}(2\pi R\cos k\delta + 2\pi R\cos(k-1)\delta)\cdot 2R\sin\tfrac{\delta}{2} \\
&= 2\pi R^2\sin\tfrac{\delta}{2}(2\cos\tfrac{1}{2}(k\delta+(k-1)\delta)\cos\tfrac{1}{2}(k\delta-(k-1)\delta)) \\
&= 2\pi R^2(2\sin\tfrac{\delta}{2}\cos\tfrac{\delta}{2})\cos\tfrac{(2k-1)\delta}{2} \\
&= 2\pi R^2\sin\delta\cos\tfrac{(2k-1)\delta}{2}
\end{aligned}
$$

Now we know that $\frac{\sin 2\theta}{2\sin\theta} = \cos\theta$, and if $\sum_{r=1}^{k}\cos(2r-1)\theta = \frac{\sin 2k\theta}{2\sin\theta}$, then $\sum_{r=1}^{k+1}\cos(2r-1)\theta = \frac{\sin 2k\theta}{2\sin\theta} + \cos(2k+1)\theta = \frac{\sin 2(k+1)\theta}{2\sin\theta}$, so by induction, for $n \geq 1$, $\sum_{r=1}^{n}\cos(2r-1)\theta = \frac{\sin 2n\theta}{2\sin\theta}$, so with $\theta = \frac{\delta}{2}$ and noting that $n\delta = \frac{\pi}{2}$ we have that if S is the surface formed by all the S_k and A is the area of S, then

$$
\begin{aligned}
A &= 2\pi R^2\sin\delta\textstyle\sum_{r=1}^{n}\cos\frac{(2r-1)\delta}{2} \\
&= 2\pi R^2\sin\delta\cdot\frac{\sin(2n\cdot\frac{\delta}{2})}{2\sin\frac{\delta}{2}} \\
&= 2\pi R^2\cos\tfrac{\delta}{2}\sin n\delta \\
&= 2\pi R^2\cos\tfrac{\pi}{4n}
\end{aligned}
$$

Hence the surface area of a sphere is $2\times\lim\limits_{n\to\infty} A = 4\pi R^2\cos 0 = 4\pi R^2$