http://www4.tpgi.com.au/nanahcub/2004hsc8bsol.pdf

NSW HSC Mathematics Extension 2 Examination 2004

Question 8. (b) Solution*

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(i) $I_n + I_{n+2} = \int_0^{\pi/4} \tan^n x\,dx + \int_0^{\pi/4} \tan^n x \tan^2 x\,dx$
$= \int_0^{\pi/4} \tan^2 x\,dx + \int_0^{\pi/4} \tan^n x(\sec^2 x - 1)\,dx$
$= \int_0^{\pi/4} \tan^n x \sec^2 x\,dx$
$= \frac{1}{n+1}[\tan^{n+1} x]_0^{\pi/4}$
$= \frac{1}{n+1} - 0$
$= \frac{1}{n+1}$

(ii) $J_n - J_{n-1} = (-1)^n I_{2n} - (-1)^{n-1} I_{2(n-1)}$
$= (-1)^n (I_{2n-2} + I_{2n-2+2})$
$= (-1)^n \cdot \frac{1}{2n-2+1}$
$= \frac{(-1)^n}{2n-1}$ for $n \geq 1$.

(iii) $J_m = (J_m - J_{m-1}) + (J_{m-1} - J_{m-2}) + (J_{m-2} - J_{m-3}) + \cdots + (J_1 - J_0) + J_0$
$= J_0 + \sum_{n=1}^m (J_n - J_{n-1})$
$= (-1)^0 I_{2(0)} + \sum_{n=1}^m \frac{(-1)^n}{2n-1}$
$= I_0 + \sum_{n=1}^m \frac{(-1)^n}{2n-1}$
$= \int_0^{\pi/4} dx + \sum_{n=1}^m \frac{(-1)^n}{2n-1}$
$= \frac{\pi}{4} + \sum_{n=1}^m \frac{(-1)^n}{2n-1}$

(iv) $u = \tan x \Rightarrow du = \sec^2 x\,dx = (1 + \tan^2 x)\,dx = (1 + u^2)\,dx \therefore dx = \frac{du}{1+u^2}$ & $\tan \frac{\pi}{4} = 1$ & $\tan 0 = 0 \therefore I_n = \int_0^1 \frac{u^n}{1+u^2}\,du$.

(v) Both I_n & I_{n+2} are positive numbers. They add to give $\frac{1}{n+1}$ & are therefore both less than $\frac{1}{n+1}$. $\therefore 0 \leq I_n \leq \frac{1}{n+1}$. As $n \to \infty, \frac{1}{n+1} \to 0$. $\therefore I_n \to 0,\ I_{2n} \to 0$ & $J_n = (-1)^n I_{2n} \to 0$ (& also $\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$)

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*The question paper is available at http://gianthole.tripod.com/exams/2004hsc.pdf