<u>1916 Leaving Certificate, 1989 and 2001 HSCs - and other niceties.</u>

4 Unit 1989 HSC Q8b

Zero candidates in 1989 got it out and that year had one of the largest candidatures for 4 unit in it's entire history, largely due to the super-generous scaling policy for that year for 4 unit maths. The examiners got back at the 4 unit cohort (and the Technical Scaling Committee) by putting this question in the exam.

James Coroneos (by far the most prolific of all hsc maths authors) had this to say about it:

> ``This question 8(b) was, in my opinion, far too long and too difficult to be included in the paper as a fair test of the 4 Unit Course."

If you can do the 1989 4unit HSC Q8b question, you should do well in any Extension 2 exam, except perhaps the incorrect 2001 paper.

During the 2001 hsc mathematics extension 2 examination, examiners informed most students that question 7 (b) (ii) was wrong and instructed them to correct it. But some students were not informed about this and their paper remained uncorrected, thereby disadvantaging some students. The following question which, although correct, students who had not corrected 7 (b) (ii) would also have been disadvantaged in by virtue of lost time.

The Masters Review was supposed to have addressed this issue, but passed it off as a mere typographical error. That's OK for some who had it corrected, but what about those who didn't? How can you compare the marks of those who corrected it to those who didn't? You can't! This not mentioned in the Masters Review.

Here is the official Board of Studies' response to this problem:

``All Mathematics Extension 2 candidates received the same examination paper. Question 7(b)(ii) contained a typographical error, and Presiding Officers were asked to make an announcement to candidates, instructing them to correct this error before the commencement of the examination. The Supervisor of Marking was informed that students from several centres may not have made this correction to their papers. He instructed the marking team for this question to pay particular attention to whether any candidate displayed any evidence of being disadvantaged by the error on the paper, and to draw such scripts to his attention. The Supervisor of Marking was also asked to monitor the marking of this question for the centres affected, and to provide a report to the Director, Examinations and Certification concerning each student's performance on Question 7(b)(ii). Appropriate compensation will be implemented to ensure that no student is disadvantaged." - Board of Studies, 2001 (which I notice is missing from the Masters Review).

Again, hoplessly inadequate and statistically invalid. The whole purpose of giving them the exam is to compare students on the basis of the same thing, not different things! Hence the 2001 exam was a complete waste of time (again not mentioned in the Masters Review).

And here is Ty Webb's Daily Telegraph article dated Monday, November 19, 2001, E-talk, page 23:

``According to your editorial headed ``Academic standards on target" (Daily Telegraph, November 17), all HSC students will be marked in the same manner, the process is straightforward and we can only trust that we have taught them well. How can they be marked in the same manner when their papers contain mistakes and only half the candidates are instructed to correct them? When this happens, there is no way you can call it a straightforward process. Some of us may have taught some of them well, but they have all been examined extremely unwell indeed."

That's more like it! (And it goes without saying that this is not in the Masters Review!)

The solutions to 7 (b) and 8 can be found appended to this note. The latter contains 4 different proofs of the irrationality of e. Also appended is a proof of the irrationality of pi. Another proof of the irrationality of pi is the simple observation that all transcendental numbers are irrational, so since pi is transcendental, it must ipso facto be irrational. That pi is transcendental is however not a simple observation. A consequence of this is that a square of side length equal to the circumference of a given circle cannot be constructed by use of ruler and compasses alone, an ancient problem nobody has been able to solve since we now know through the transcendentality of pi that such a construction is entirely impossible. Such a construction is however possible by use of such niceties as the Spiral of Archimedes, but that circumvents the problem rather than solves it since such niceties cannot themselves be constructed with ruler and compasses alone! Why the preponderance on rulers and compasses? Because anything you do with them will change lengths by multiples of algebraic numbers, not transcendental ones. And here we unexpectedly find ourselves catapulted onto the more exalted plane of mathematical discourse of the not well understood connection between the two great bastions of The Old School of mathematics: geometry and number theory, but also a beacon of future mathematical enlightenment. Therein lies the seeds of further contentment such as the proof of Fermat's Last Theorem - an epitome of greatness indeed. All's well that ends well - I suppose. Just don't drink all your Grange Hermitage and Hill of Grace because the Riemann Hypothesis still hasn't been proved. (At least it's better reading than the Masters Review!)

Those advocating that past papers get out of date the older they are and hence the older ones are not worth studying to prepare for exams should pause to reflect that Question 7 (a) in the 2001 Mathematics Extension 2 paper is almost identical to Question 12 in the 1916 Leaving Certificate Mathematics Honours Paper I, the oldest externally examined Leaving Certificate! Of course the 2001 examiners knew that because that's where they got it from in the first place, but they didn't think that anybody would ever find out. Alas this webpage seeks to humble the exalted and to exalt the humble! Just who is it that advocates that looking at old papers is a pointless exercise? Perhaps I'll put up the Leaving Certificate papers 1917-1966 at some later stage onto the internet for you, much to the behest of examiners! I'm sure you will agree though that the 1916 Seppelt Para Liqueur is a good drop and is still holding up well (yet another thing not mentioned in the Masters Review).

1916 New South Wales Leaving Certificate

Mathematics Honours Papers

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

New South Wales Department of Education

PAPER I

Time Allowed: 3 Hours

1. Prove that the radical axes of three circles taken in pairs are concurrent. Use this theorem to obtain a construction for a circle through two given points which shall touch a given circle. Prove the validity of your construction, and show that there are two such circles.

2. Prove that the inverse of a circle with regard to a point upon its circumference is a straight line, and with regard to a point not on its circumference is another circle. Two parallel straight lines lie on the same side of a given point P. A circle is described to pass through P and to touch the more remote of the given lines at Z, cutting the other at X and Y. Prove that the angles XPZ and YPZ are equal. Draw the system of lines and circles which you obtain by inverting the system with regard to P, and enunciate the inverse proposition.

3. A, B, C, D are the four corners of a face of a cube, taken in order, and $CDEF$ is another face of the same cube, AED and BCF being the faces perpendicular to $CDEF$. Find the sizes of the angles AEF, AFE, DFE, and AFD. Also find the length of the perpendicular from E on ADF.

4. Prove the two following theorems:-

(i) Of the three plane angle which form a trihedral angle, any two are together greater than the third.

(ii) The sum of the plane angles of a convex solid angle is less than four right angles.

5. With the usual notation prove that in the parabola

$$PN^2 = 4AS.AN$$

SL is the semi-latus rectum of a parabola, whose vertex is A, and X is the point where the axis meets the directrix. A second parabola has S for its focus and X for its vertex. The line LX meets the directrix of this second parbola in Q and QP is drawn parallel to the axis to meet the same parbola in P. Show that the ordinate of P bisects AX.

6. Prove, by orthogonal projection of a circle, the following properties of an ellipse:-

(i) The tangents at the ends of any chord meet on the diameter which bisects the chord.

(ii) If a diameter CP meets a chord in V, and the tangents at its extremities in T, then $CV.CT = CP^2$.

(iii) Lines drawn through any point of an ellipse to the extremities of any diameter meet the conjugate diameter CD in M and N. Show that

$$CM.CN = CD^2.$$

7. Prove that

$$\cos A = 4\cos^3 \tfrac{A}{3} - 3\cos \tfrac{A}{3},$$
$$\sin A = 3\sin \tfrac{A}{3} - 4\sin^3 \tfrac{A}{3}.$$

(i) Find all the possible values of $\cos \frac{A}{3}$.

(ii) Find all the possible values of $\sin \frac{A}{3}$.

(iii) Find how many values there are in general for $\sin \frac{A}{3}$ and for $\cos \frac{A}{3}$.

8. Prove that if $\cos 2\theta = \cos 2\alpha \cos 2\beta$, then $\sqrt{(\cot^2 \alpha + \cot^2 \beta)} = \frac{\sin\theta}{\sin\alpha \sin\beta}$. If $\alpha = 62°$ and $\beta = 54°$, find all the possible values of θ which satisfy the former of these equations, and show that their sines have the same absolute value.

The station A is due south, and the station B due east, of O. The altitudes of an aeroplane vertically above O are $62°$ and $54°$ from A and B respectively. If AB is 1 mile, find the height of the aeroplane above O.

9. R is the radius of the circumcircle and r that of the inscribed circle of the triangle ABC. Prove that

$$r = 4R \sin \tfrac{A}{2} \sin \tfrac{B}{2} \sin \tfrac{C}{2}.$$

The inscribed circle touches the sides of the triangle ABC in D, E, F. Show that the area of the triangle DEF is given by $Rr \sin A \sin B \sin C$.

10. Prove that, for real values of $n, \cos n\theta + i \sin n\theta$ is the value, or one of the values, of $(\cos\theta + i\sin\theta)^n$, and in cases in which there is more than one value give all the values of $(\cos\theta + i\sin\theta)^n$. Find a linear factor and two quadratic factors of $x^5 + 1$.

11. What is a radian? Show that π radians are equal to 2 right angles. Verify that between $108°$ and $109°$ there is an angle whose sine is half its circular measure. Show, graphically or otherwise, that there is only one positive angle with this property.

12. Sum to n terms the following series:-

(i) $\cos\theta + \cos 2\theta + \cos 3\theta + \cdots$

(ii) $\sin\theta + \sin 2\theta + \sin 3\theta + \cdots$

(iii) $\cos\theta + 2\cos 2\theta + 3\cos 3\theta + \cdots$

(iv) $\sin\theta + 2\sin 2\theta + 3\sin 3\theta + \cdots$

Also show that the series

$$\cos\theta + x\cos 2\theta + x^2\cos 3\theta + \cdots$$

is convergent when $-1 < x < 1$, and that its sum is

$$\frac{\cos\theta - x}{1 - 2x\cos\theta + x^2}$$

END OF PAPER I

PAPER II BEGINS ON THE NEXT PAGE

PAPER II

Time Allowed: 3 Hours

1. What is the ratio test for the convergence of the series

$$u_0 + u_1 + u_2 + \cdots \quad ?$$

Establish the theorem in the form in which you have stated it. Discuss the convergence of the series

(i) $1 + nx + \frac{n(n-1)}{1\cdot 2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \cdots \quad (-1 < x < 1)$

(ii) $x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \quad (-1 < x < 1).$

Show that the second series is convergent when $x = 1$.

2. Apply the Binomial Theorem to show that

$$\left(\tfrac{3}{4}\right)^{\frac{4}{5}} = \cdot 7944$$

correct to four decimals. Establish carefully that your approximation is as stated.

3. Solve the following question in permutations and combinations:-

(a) Six persons A, B, C, D, E, and F are to address a meeting. In how many ways can they take turns so that

(i) A, B are consecutive speakers;

(ii) A, B are consecutive speakers, and C, D are consecutive speakers?

The order of A and B respectively, or C and D respectively, is not to be considered.

(b) There are six persons from whom a game of tennis, two on each side, is to be made up. How many different matches could be arranged, a change in either pair giving a different match?

4. Establish the identity

$$\frac{x}{x^5+1} = -\frac{1}{5(x+1)} - \frac{2}{5}\sum_{r=1}^{2} \frac{x\cos 2(2r-1)\frac{\pi}{5} - \cos(2r-1)\frac{\pi}{5}}{x^2 - 2x\cos(2r-1)\frac{\pi}{5} + 1}$$

5. Assuming that

$$\log_e(1+x) = x - \tfrac{x^2}{2} + \tfrac{x^3}{3} - \cdots, \quad (-1 < x < 1)$$

show that

$$\log_e\left(\tfrac{1+x}{1-x}\right) = 2\left(x + \tfrac{x^3}{3} + \cdots\right),$$

when x is numerically less than unity. Also show that the error in taking

$$2\left(x + \tfrac{x^3}{3} + \cdots + \tfrac{x^{2n-1}}{2n-1}\right)$$

for $\log_e\left(\tfrac{1+x}{1-x}\right)$ is less in absolute value than

$$\tfrac{2|x|^{2n+1}}{(2n+1)(1-x^2)},$$

with the same range for x.

6. Use the second part of the preceding question to find $\log_e 2$ and $\log_e 10$ correct to seven places.

7. Find the condition that the three points

$$(x_1, y_1), (x_2, y_2), (x_3, y_3)$$

may be collinear. A is the point $(a, 0)$, B the point $(0, b)$ and C the point (b, a). AC meets the axis of y in F, and BC meets the axis of x in E. Find the co-ordinates of E and F; also show, by analytical geometry, that the middle points of EF, OC and AB are collinear.

8. Find the equation of the line through (x_0, y_0) perpendicular to the line $ax + by + c = 0$. Show that the distance from (x_0, y_0) to the point of intersection of these two lines is

$$\frac{(ax_0 + by_0 + c)}{\sqrt{(a^2 + b^2)}}.$$

Verify that the point $(3\sqrt{5} - 7, \sqrt{5} + 1)$ is equally distant from the three lines

$$\left.\begin{aligned} x + 2y - 10 &= 0 \\ 4x + 3y \quad\;\; &= 0 \\ 2x - \;\, y \quad\;\; &= 0 \end{aligned}\right\}.$$

9. Prove that the equation of the tangent at the point $(c\cos\theta, c\sin\theta)$ on the circle $x^2 + y^2 = c^2$ is

$$x\cos\theta + y\sin\theta = c.$$

Show that if θ satisfies the equation

$$a\cos\theta + b\sin\theta = c,$$

the tangent at $(c\cos\theta, c\sin\theta)$ passes through the point (a, b), and thus obtain a graphical solution of this equation, when $c^2 < a^2 + b^2$. By means of this figure, show that, if α, β are the roots of the equation, then

$$\frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{c} = \frac{\cos\left(\frac{\alpha+\beta}{2}\right)}{a} = \frac{\sin\left(\frac{\alpha+\beta}{2}\right)}{b}$$

10. Find the centre and radius of each of the circles:-

$$\left.\begin{aligned} x^2 + y^2 - 4x - 2y + 1 = 0 \\ x^2 + y^2 - 5x + y - 6 = 0 \end{aligned}\right\}.$$

Draw the circles and thus obtain graphically the values of (x, y) which satisfy both equations. Find the equation of the common chord of the circles, and show that it cuts the axis of y at a distance $\frac{7}{3}$ from the origin.

11. Find, from the definition of the differential coefficient, the value of $\frac{dy}{dx}$, when $y = (x-1)(x-2)$. Show that the gradients of this curve at the points where $x = 1, x = \frac{3}{2}$ and $x = 2$ are respectively $-1, 0$ and $+1$. Obtain the equations of the tangents at these points.

12. Prove the rule for differentiating the product of two functions. Differentiate the expression $(ax+b)^2(cx+d)^2$ in two ways, first as a product, then after multiplying it out. Show that the gradient of the curve

$$y = (ax+b)^2(cx+d)^2$$

is zero, when x has the values $-\frac{b}{a}, -\frac{d}{c}$, and $-\frac{ad+bc}{2ac}$.

13. Find an expression for the area of a parabola from the vertex to the latus rectum. Two parabolas have the same focus and axis, and their concavities are turned in the same direction. If the vertices are distant a and b from the focus, show that the areas cut from each by the latus rectum are in the ratio $a^2 : b^2$.

14. Prove that the volume V of a solid of revolution, the axis of y being the axis of the solid, satisfies the equation

$$\frac{dV}{dy} = \pi x^2.$$

A hemispherical bowl of radius a ft. stands with its axis vertical, and a solid cone of semi-vertical angle α is placed in the bowl with its vertex at the lowest point and its axis also vertical. If the space between the cone and the bowl is to be filled with water to a depth h ft., show that $\pi h^2(a - \frac{h}{3}\sec^2\alpha)$ cub. ft. will be required.

NSW HSC 4 Unit Mathematics Examination 1989 Question 8(b)

+ Solution

8(b). The difference between a real number r are the greatest integer less than or equal to r is called the fractional part of $r, F(r)$. Thus $F(3.45) = 0.45$. Note that for all real numbers $r, 0 \leqq F(r) < 1$.

(i) Let $a = 2136 \log_{10} 2$.

Given that $\quad F(a) = \ \ 7.0738 \cdots \times 10^{-5}$

observe that $\quad F(2a) = 14.1476 \cdots \times 10^{-5}$

$$F(3a) = 21.2214 \cdots \times 10^{-5}$$

(α) Use your calculator to show that $\log_{10} 1.989 < F(4223a) < \log_{10} 1.990$.

(β) Hence calculate an integer M such that the ordinary decimal representation of 2^M begins with 1989. Thus 2^M=1989 $\ldots$.

(ii) Let r be a real number and let m and n be non-zero integers with $m \neq n$.

(α) Show that if $F(mr) = 0$, then r is rational.

(β) Show that if $F(mr) = F(nr)$, then r is rational.

(iii) Suppose that b is an irrational number. Let N be a positive integer and consider the fractional parts $F(b), F(2b), \ldots, F((N+1)b)$.

(α) Show that these $N+1$ numbers $F(b), \ldots, F((N+1)b)$ are all distinct.

(β) Divide the interval $0 \leqq x < 1$ into N subintervals each of length $1/N$ and show that there must be integers m and n with $m \neq n$ and $1 \leqq m,\ n \leqq N+1$ such that $F((m-n)b) < 1/N$.

(iv) Given that $\log_{10} 2$ is irrational, choose any integer N such that $1/N < \log_{10} \frac{1990}{1989}$; note that in **(i)** , $F(a) < \log_{10} \frac{1990}{1989}$.

Use **(iii)** to decide whether there exists another integer M such that $2^M = 1989 \ldots$.

Solution.
Let $r \in \mathbb{R}$ and let $F(r)$ denote the fractional part of r and $\lfloor r \rfloor$ denote the integer part of r, i.e., $\lfloor r \rfloor = r - F(r)$. Let $a = 2136 \log_{10} 2$.

(i)(*α*) For positive integers n, $F(na) = nF(a)$ for $1 \leq n \leq \left\lfloor \frac{1}{F(a)} \right\rfloor = 14136$, so

$$\begin{aligned}
\log_{10} 1.989 &= 0.29863478\ldots \\
&< 0.29872657\ldots \\
&= 4223F(a) \\
&= F(4223a) \text{ since } 1 \leq 4223 \leq 14136 \\
&< 0.29885307\ldots \\
&= \log_{10} 1.990.
\end{aligned}$$

Hence $\log_{10} 1.989 < F(4223a) < \log_{10} 1.990$.

(*β*) From **(*α*)** since $\phi : \mathbb{R} \to \mathbb{R}^+ : x \mapsto 10^x$ is a strictly increasing function of x,

$$\begin{aligned}
1.989 < &\quad 10^{F(4223a)} && < 1.990 \\
1.989 < &\quad 10^{(\lfloor 4223a \rfloor + F(4223a)) - \lfloor 4223a \rfloor} && < 1.990 \\
1989 \times 10^{\lfloor 4223a \rfloor - 3} < &\quad 10^{4223a} = 2^{\log_2(10^{4223a})} && \\
&\quad = 2^{4223a/\log_{10} 2} && \\
&\quad = 2^{4223 \times 2136} && \\
&\quad = 2^{9020308} && < 1990 \times 10^{\lfloor 4223a \rfloor - 3} \\
1989 \times 10^{2715386} < &\quad 2^{9020308} && < 1990 \times 10^{2715386}
\end{aligned}$$

Hence an integer M such that the ordinary decimal representation of 2^M begins with 1989, thus $2^M = 1989\ldots$, is $M = 9020308$.

(ii)(*α*) $(r \in \mathbb{R}, m \in \mathbb{Z} : m \neq 0 \;\&\; F(mr) = 0) \Rightarrow mr - \lfloor mr \rfloor = 0 \Rightarrow r = \frac{\lfloor mr \rfloor}{m} \in \mathbb{Q}$.

(*β*) $(r \in \mathbb{R}, m, n \in \mathbb{Z} : 0 \neq m \neq n \neq 0 \;\&\; F(mr) = F(nr)) \Rightarrow$
$mr - \lfloor mr \rfloor = nr - \lfloor nr \rfloor \Rightarrow r = \frac{\lfloor mr \rfloor - \lfloor nr \rfloor}{m-n} \in \mathbb{Q}$.

(iii) Let $b \in \mathbb{R} \backslash \mathbb{Q}$ and $N \in \mathbb{Z}^+$.

(*α*) Let $m, n \in \mathbb{Z} : 0 \neq m \neq n \neq 0 \;\&\; 1 \leq m, n \leq N+1$.
Then $(F(mb) = F(nb)) \Rightarrow b \in \mathbb{Q}$ from **(ii)(*β*)**, but $b \in \mathbb{R}\backslash\mathbb{Q}$ - contradiction.
Hence $F(mb) \neq F(nb)$, so $F(b), F(2b), \ldots, F((N+1)b)$ are distinct.

(*β*) Since $F(b), \ldots, F((N+1)b)$ are $N+1$ distinct real numbers from **(iii)(*α*)**, in the interval $0 \leq x < 1$, which can be subdivided into N subintervals, each of length $\frac{1}{N}$, then at least 2 of these real numbers must lie in the same subinterval distant less than $\frac{1}{N}$ apart. Let $F(mb), F(nb)$ be 2 such real numbers where the integers m, n are such that $m \neq n, 1 \leq m, n \leq N+1$, and without loss of generality, $1 > F(mb) > F(nb) > 0$ since b is irrational. Then

$$\begin{aligned} F((m-n)b) &= F(mb - nb) \\ &= F((\lfloor mb \rfloor + F(mb)) - (\lfloor nb \rfloor + F(nb))) \\ &= F((\lfloor mb \rfloor - \lfloor nb \rfloor) + (F(mb) - F(nb))) \\ &= F(F(mb) - F(nb)) \\ &= F(mb) - F(nb) \text{ since } 0 < F(nb) < F(mb) < 1 \\ &< \tfrac{1}{N} \text{ since } F(mb) \,\&\, F(nb) \text{ are distant less than } \tfrac{1}{N} \text{ apart.} \end{aligned}$$

Hence there must be integers m and n with $m \neq n$ and $1 \leq m, n \leq N+1$ such that $F((m-n)b) < 1/N$.

(iv) $F(a) < \frac{1}{N} < \log_{10} \frac{1990}{1989}$ for $\left\lfloor \frac{1}{\log_{10} \frac{1990}{1989}} \right\rfloor + 1 = 4581 \leq N \leq \left\lfloor \frac{1}{F(a)} \right\rfloor = 14136$ & $\therefore$ since $4581 < 10^4 < 14136$, then $F(a) < \frac{1}{N} < \log_{10} \frac{1990}{1989}$ when $N = 10^4$.

For positive integers n, $F(na) = nF(a)$ for $1 \leq n \leq \left\lfloor \frac{1}{F(a)} \right\rfloor = 14136$, and $1 \leq 4223 \leq 4224 \leq 14136$. So

$$\begin{aligned} \log_{10} 1.989 &< \frac{\lfloor 10^4 \log_{10} 1.989 \rfloor + 1}{10^4} \\ &= \frac{2987}{10^4} \\ &= \frac{\lfloor 10^4 F(4223a) \rfloor}{10^4} \\ &< \frac{10^4 F(4223a)}{10^4} \\ &= F(4223a) \\ &= 4223F(a) \\ &< 4224F(a) \\ &= F(4224a) \\ &< \frac{\lfloor 10^4 F(4224a) \rfloor + 1}{10^4} \\ &= \frac{2988}{10^4} \\ &= \frac{\lfloor 10^4 \log_{10} 1.990 \rfloor}{10^4} \\ &< \frac{10^4 \log_{10} 1.990}{10^4} \\ &= \log_{10} 1.990. \end{aligned}$$

Since $\log_{10} 2$ is irrational, then so is any nonzero integer multiple of $\log_{10} 2$ and so a is irrational. So using **(iii)($\boldsymbol{\beta}$)** with $a = b, N = 10^4, m = 4224, n = 4223, F(ma) > F(na)$ & $F(ma) - F(na) = F((m-n)a) = F(a) < \frac{1}{N} < \log_{10} \frac{1990}{1989}$ & $F(ma)$ & $F(na)$ lie strictly between $\frac{2987}{N}$ and $\frac{2988}{N}$ which lie strictly between $\log_{10} 1.989$ and $\log_{10} 1.990$, i.e.,

$$\log_{10} 1.989 < \frac{2987}{10^4} < F(4223a) < F(4224a) < \frac{2988}{10^4} < \log_{10} 1.990$$

Hence from **(i)($\boldsymbol{\beta}$)**, there exists another integer $M = ma/\log_{10} 2 = 4224 \times 2136 = 9022464$ such that $2^M = 1989\ldots$.

Moreover,

Let $t_0 = 4223, t_i = t_{i-1} + \lfloor 1/F(a) \rfloor - \lfloor F((t_{i-1} + \lfloor 1/F(a) \rfloor)a) - F(t_0 a) \rfloor$ for $i = 1, 2, \ldots$. From **(iii)($\boldsymbol{\alpha}$)**, since a is irrational, then for any $N > 0$, $F(a), F(2a), \ldots, F((N+1)a)$ are distinct, then $F(a), F(2a), \ldots, F((N+1)a), \ldots$ are all distinct & $\therefore$ $F(t_i a) \neq F(t_j a)$ for all i, j such that $i \neq j$ & for all $i > 0$,

$$\begin{aligned}\log_{10} 1{\cdot}989 &= 0{\cdot}29863478\ldots \\ &< 0{\cdot}29872657\ldots \\ &= F(t_0 a) \\ &< F(t_i a) \\ &< F((t_0+1)a) \\ &= 0{\cdot}29879731\ldots \\ &< 0{\cdot}29885307\ldots \\ &= \log_{10} 1{\cdot}990\end{aligned}$$

Hence it follows from **(iii)($\boldsymbol{\alpha}$)** that there are infinitely many integers t_i such that $\log_{10} 1{\cdot}989 < F(t_i a) < \log_{10} 1{\cdot}990$, & $\therefore$ from **(i)($\boldsymbol{\beta}$)**, that there are infinitely many integers $M = 2136 t_i$ such that $2^M = 1989\ldots$.

Note that if the restriction that we must use **(iii)** to solve **(iv)** is removed, then it would be much easier because $2^{290} =$
1989292945639146568621528992587283360401824603189390869761855907572637988050133502132224.

2001 HSC Mathematics Extension 2 Q7b + solution

Up until Question 7 (of 8) the 2001 HSC Mathematics Extension 2 paper was OK. (Tutts, P., 2001)

Some students will have found this quite tough but that's because they expect a lot of themselves and when they can't answer every question it worries them - but that's only to be expected. (Vanderhout, N. and Mealey, L., 2001)

Well up until Question 7, the paper was correct. It's quite tough in mathematics to prove incorrect statements, in fact it's so tough it's impossible! It worries me that the examination would ask students to do the impossible - but that's only to be expected when examiners don't correct the paper.

I usually don't put solutions on the internet. But there are exceptions, and this is one such exception, not for anything in the question, but rather for the circumstances under which it was examined.

The examiners in most schools instructed students to change the original formula

$$3r^3 s + s^3 d - 3s = 0 \cdots \times$$

to

$$3r^2 s + s^3 d - 3s = 0 \cdots \checkmark$$

for **Question 7 (b) (ii)**, yet examiners in some schools did not instruct students to change it.

So some students were undoubtably disadvantaged by this problem.

The Board of Studies have decided to mark students who were not told to change it out of a smaller total, ignoring **7 (b) (ii)**. But this is not a statistically valid way to solve the problem.

The corrected question is on the next page.

7(b) Consider the equation $x^3 - 3x - 1 = 0$, which we denote by (*).

(i) Let $x = \frac{p}{q}$ where p and q are integers having no common divisors other than $+1$ and -1. Suppose that x is a root of the equation $ax^3 - 3x + b = 0$, where a and b are integers.
Explain why p divides b and why q divides a. Deduce that (*) does not have a rational root.

(ii) Suppose that r, s and d are rational numbers and that $\sqrt{d}$ is irrational. Assume that $r + s\sqrt{d}$ is a root of (*).

Show that $3r^2s + s^3d - 3s = 0$ and show that $r - s\sqrt{d}$ must also be a root of (*).

Deduce from this result and part **(i)**, that no root of (*) can be expressed in the form $r + s\sqrt{d}$ with r, s and d rational.

(iii) Show that one root of (*) is $2\cos\frac{\pi}{9}$.

(You may assume the identity $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.)

(NSW Board of Studies, 2001)

The solution starts on the next page.

Solution.

7 (b) (i) $\left(x-\frac{p}{q}\right)\left(ax^2+\frac{ap}{q}x+\frac{ap^2}{q^2}-3\right)=ax^3-3x+\frac{p(3q^2-ap^2)}{q^3}=ax^3-3x+b$

$\Rightarrow b=\frac{p(3q^2-ap^2)}{q^3}$

$\Rightarrow q^3b=p(3q^2-ap^2)=q^2(3p)-ap^3$ .... (†)

p,q coprime, $p|p(3q^2-ap^2),(\dagger)\Rightarrow p|q^3b\Rightarrow p|b$.

p,q coprime, $q|q^3b, q|q^2(3p),(\dagger)\Rightarrow q|ap^3\Rightarrow q|a$.

Suppose (*) had a rational root, say $x=\frac{p}{q}$ for coprime p and q.

Then $a=1, b=-1\Rightarrow p|-1\ \&\ q|1\Rightarrow p=1$ or -1 and $q=1$ or -1

$\Rightarrow x=\pm1\Rightarrow(\pm1)^3-3(\pm1)-1=0\Rightarrow -3=0$ or $1=0$

$\Rightarrow (*)$ does not have a rational root $\square$

(ii) $r,s,d\in\mathbb{Q},\sqrt{d}\notin\mathbb{Q}, r+s\sqrt{d}$ is a root of (*) $\Rightarrow$

$(r+s\sqrt{d})^3-3(r+s\sqrt{d})-1=r^3+3r^2s\sqrt{d}+3rs^2d+s^3d\sqrt{d}-3r-3s\sqrt{d}-1=0$

$\Rightarrow (r^3+3rs^2d-3r-1)+(3r^2s+s^3d-3s)\sqrt{d}=0$

$\Rightarrow r^3+3rs^2d-3r-1=3r^2s+s^3d-3s=0$ for otherwise,

$\sqrt{d}=\frac{-(r^3+3rs^2d-3r-1)}{3r^2s+s^3d-3s}\in\mathbb{Q}$ - contradiction.

Hence $3r^2s+s^3d-3s=0$.

$s\neq0$ for otherwise (*) would have a rational root, which by virtue of **(i)** it doesn't. So we can divide by s:

$3r^2+s^2d-3=0$.

$\therefore -3r^2-s^2d=-3$.

and $3r^3+rs^2d-3r=0$ multiplying through by r.

So $-r^3-3rs^2d+3r=-1=2r^3-2rs^2d$.

$$\begin{aligned}\text{Hence } x^3-3x-1&=x^3+(-3r^2-s^2d)x+2r^3-2rs^2d\\&=(x^2-2rx+r^2-s^2d)(x+2r)\\&=(x-r-s\sqrt{d})(x-r+s\sqrt{d})(x+2r).\end{aligned}$$

$\therefore r - s\sqrt{d}$ is a root of (*). But then so is $-2r \in \mathbb{Q}$ a root of (*). But **(i)**$\Rightarrow$(*) has no such root. Therefore (*) has no root in the form $r + s\sqrt{d}$ either (nor indeed of the form $r - s\sqrt{d}$) with r, s and d rational with $\sqrt{d}$ irrational.

Moreover, if $\sqrt{d}$ is rational, so is $r + s\sqrt{d}$ and so by virtue of **(i)** in this case also, $r + s\sqrt{d}$ could not be a root of (*).

Hence more generally, no root of (*) can be expressed in the form $r + s\sqrt{d}$ with $r, s, d \in \mathbb{Q}$ $\square$

(iii) Let $x = u - v$. Then

$$\begin{aligned}(u-v)^3 - 3(u-v) - 1 &= u^3 - v^3 - 1 - (3uv(u-v) + 3(u-v))\\ &= u^3 - v^3 - 1 - (u-v)(3uv+3)\\ &= 0\end{aligned}$$

where u and v are such that $3uv + 3 = 0$ so $uv = -1$ and $u^3v^3 = (uv)^3 = -1$.

$$\begin{aligned}\text{Now } u^3 - v^3 - 1 &= \tfrac{u^3(u^3-v^3-1)}{u^3}\\ &= \tfrac{u^6-u^3v^3-u^3}{u^3}\\ &= \tfrac{1}{u^3}(u^6 - u^3 + 1)\\ &= \tfrac{1}{u^3}((u^3)^2 - u^3 + 1)\\ &= 0\end{aligned}$$

$$\Rightarrow u^3 = \tfrac{1\pm\sqrt{1-4}}{2} = \tfrac{1\pm\sqrt{3}i}{2} = e^{\pm\pi i/3} = e^{(6k\pm1)\pi i/3} \text{ for } k = -1, 0, 1$$

and $v^3 = -\frac{1}{u^3} = -e^{(-6k\mp1)\pi i/3}$ where $k = -1, 0, 1$

$$\begin{aligned}\Rightarrow x &= e^{(6k\pm1)\pi i/9} + e^{(-6k\mp1)\pi i/9} \text{ for } k = -1, 0, 1\\ &= 2\cos\tfrac{n\pi}{9} \text{ for } n = 1, 5, 7 \textbf{ assuming nothing}\end{aligned}$$

are 3 distinct roots of (*) which has degree 3, so these are all the roots of (*) by the Fundamental Theorem of Algebra.

In particular, $2\cos\frac{\pi}{9}$ is one of the roots of (*) $\square$

Alternatively,

$$\begin{aligned}\left(2\cos\tfrac{\pi}{9}\right)^3 - 3\left(2\cos\tfrac{\pi}{9}\right) - 1 &= 8\cos^3\tfrac{\pi}{9} - 6\cos\tfrac{\pi}{9} - 1\\ &= 2\left(4\cos^3\tfrac{\pi}{9} - 3\cos\tfrac{\pi}{9}\right) - 1\\ &= 2\cos\tfrac{\pi}{3} - 1 \text{ \textbf{assuming} } \boldsymbol{\cos 3\theta = 4\cos^3\theta - 3\cos\theta}\\ &\qquad\qquad \text{\textbf{where} } \boldsymbol{\theta = \tfrac{\pi}{9}}\\ &= 2\cdot\tfrac{1}{2} - 1\\ &= 0\end{aligned}$$

So one root of (*) is $2\cos\frac{\pi}{9}$ □

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2001 HSC Mathematics Extension 2 Q8 + solution

Question 8

(a) (i) Show that $2ab \leq a^2 + b^2$ for all real numbers a and b.

Hence deduce that $3(ab + bc + ca) \leq (a + b + c)^2$ for all real numbers a, b and c.

(ii) Suppose a, b and c are the sides of a triangle. Explain why $(b - c)^2 \leq a^2$.

Deduce that $(a + b + c)^2 \leq 4(ab + bc + ca)$.

(b) (i) Explain why, for $\alpha > 0$,

$$\int_0^1 x^\alpha e^x \, dx < \frac{3}{\alpha + 1}.$$

(You may assume $e < 3$.)

(ii) Show, by induction, that for $n = 0, 1, 2 \ldots$ there exist integers a_n and b_n such that

$$\int_0^1 x^n e^x \, dx = a_n + b_n e.$$

(iii) Suppose that r is a positive rational, so that $r = \frac{p}{q}$ where p and q are positive integers. Show that, for all integers a and b, either

$$|a + br| = 0 \text{ or } |a + br| \geq \frac{1}{q}.$$

(iv) Prove that e is irrational.

(NSW Board of Studies, 2001)

Solution starts on next page.

Solution.

(a) **(i)** For all $a, b \in \mathbb{R}, a - b \in \mathbb{R} \Rightarrow (a-b)^2 \geq 0$ & $\therefore$ $a^2 - 2ab + b^2 \geq 0$

& $\therefore$ $2ab \leq a^2 + b^2$.

Therefore $ab \leq \frac{a^2+b^2}{2}$.

For $c \in \mathbb{R} \Rightarrow$ similarly, $bc \leq \frac{b^2+c^2}{2}$ and $ca \leq \frac{c^2+a^2}{2}$ and therefore

$ab + bc + ca \leq \frac{a^2+b^2}{2} + \frac{b^2+c^2}{2} + \frac{c^2+a^2}{2} = a^2 + b^2 + c^2$

$\therefore 3(ab + bc + ca) \leq a^2 + b^2 + c^2 + 2(ab + bc + ca)$

$\therefore 3(ab + bc + ca) \leq (a + b + c)^2$ $\square$

(ii) a, b, c sides of a triangle $\Rightarrow a^2 = b^2 + c^2 - 2bc \cos A$ where A is the angle opposite the side of length a units.

But $-1 < \cos A < 1$ ($\cos A \neq \pm 1$ for otherwise we can't have a triangle)

Therefore $-2bc < 2bc \cos A < 2bc$ ($\because b, c > 0$)

Hence $2bc > -2bc \cos A > -2bc$

So $b^2 + c^2 + 2bc \geq b^2 + c^2 - 2bc \cos A > b^2 + c^2 - 2bc$

Thus $(b + c)^2 > a^2 > (b - c)^2$

$\Rightarrow (b - c)^2 \leq a^2$ (although a^2 is never $= (b - c)^2$.)

$\therefore -2bc \leq a^2 - b^2 - c^2$

$\therefore 2bc \geq b^2 + c^2 - a^2$.

Similarly, $2ab \geq a^2 + b^2 - c^2, 2ca \geq c^2 + a^2 - b^2$

$\therefore 2(ab + bc + ca) \geq b^2 + c^2 - a^2 + a^2 + b^2 - c^2 + c^2 + a^2 - b^2 = a^2 + b^2 + c^2$

$\therefore (a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca) \leq 4(ab + bc + ca)$ $\square$

(b) **(i)** Let $\alpha > 0$. Then

$$\int_0^1 x^\alpha e^x \ dx = \int_0^1 e^x \tfrac{d}{dx} \tfrac{x^{\alpha+1}}{\alpha+1} \ dx$$

$$= \left[\tfrac{x^{\alpha+1}}{\alpha+1} e^x\right]_0^1 - \int_0^1 \tfrac{x^{\alpha+1}}{\alpha+1} e^x \ dx$$

$$= \tfrac{e}{\alpha+1} - \int_0^1 \tfrac{x^{\alpha+1}}{\alpha+1} e^x \ dx$$

$$< \tfrac{e}{\alpha+1} \Big(\because \int_0^1 \tfrac{x^{\alpha+1}}{\alpha+1} e^x \ dx > 0 \because \tfrac{x^{\alpha+1}}{\alpha+1} e^x > 0 \text{ for } 0 < x < 1 \ \& \ \alpha > 0 \Big)$$

$$< \tfrac{3}{\alpha+1} \text{ since } e < 3 \quad \square$$

(ii) $\int_0^1 e^x \ dx = e = a_0 + b_0 e$ where $a_0 := 0$ and $b_0 := 1$.

If for $n \in \mathbb{Z}^+$ there exists $a_{n-1}, b_{n-1} \in \mathbb{Z}$ such that $\int_0^1 x^{n-1} e^x \ dx = a_{n-1} + b_{n-1} e$, then $\int_0^1 x^n e^x \ dx = \int_0^1 x^n \frac{de^x}{dx} \ dx = \left[x^n e^x\right]_0^1 - \int_0^1 e^x n x^{n-1} \ dx = e - n(a_{n-1} + b_{n-1} e) = -n a_{n-1} + (1 - b_{n-1}) e = a_n + b_n e$ where $a_n := -n a_{n-1} \in \mathbb{Z} \because n \in \mathbb{Z}$ and $a_{n-1} \in \mathbb{Z}$ and $b_n := 1 - b_{n-1} \in \mathbb{Z} \because b_{n-1} \in \mathbb{Z}$.

Therefore by mathematical induction for $n \in \mathbb{N}$, there exist $a_n, b_n \in \mathbb{Z}$ such that $\int_0^1 x^n e^x \ dx = a_n + b_n e \quad \square$

(iii) $a, b \in \mathbb{Z}, p, q \in \mathbb{Z}^+, r = \frac{p}{q} \Rightarrow aq + bp \in \mathbb{Z} \ \& \ a + br = \frac{aq+bp}{q}$.

$aq + bp = 0 \Rightarrow |a + br| = 0$.

$aq + bp \neq 0 \Rightarrow aq + bp \in \mathbb{Z}\backslash\{0\} \Rightarrow |aq + bp| \geq 1 \Rightarrow |a + br| \geq \frac{1}{q}$.

Hence either $|a + br| = 0$ or $|a + br| \geq \frac{1}{q} \quad \square$

(iv) I present 4 methods. There are many other ways to prove e irrational.

Method 1

Suppose $e \in \mathbb{Q}$. Then there exist $p, q \in \mathbb{Z}^+$ such that $e = \frac{p}{q}$.

Suppose $n \in \mathbb{Z}^+$ is such that $\frac{3}{n+1} < \frac{1}{q}$.

(i)$\Rightarrow \int_0^1 x^n e^x \ dx < \frac{3}{n+1} < \frac{1}{q}$.

(ii)$\Rightarrow$ there exist $a, b \in \mathbb{Z}$ such that $\int_0^1 x^n e^x \ dx = a + be$

(iii)$\Rightarrow |a + be| = 0$ or $|a + be| \geq \frac{1}{q}$.

But $\int_0^1 x^n e^x \ dx > 0 \Rightarrow a + be > 0$.

$\therefore \frac{1}{q} \leq |a + be| = a + be = \int_0^1 x^n e^x \ dx < \frac{1}{q}$

$\Rightarrow q < q$.

So reductio ad absurdum, we have that $e \notin \mathbb{Q}$ $\square$

Method 2

Lemma. $\lim\limits_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$.

Proof.

$$
\begin{aligned}
\ln \lim_{n\to\infty} \left(1+\tfrac{x}{n}\right)^n &= \lim_{n\to\infty} \ln \left(1+\tfrac{x}{n}\right)^n \\
&= \lim_{n\to\infty} n \ln \left(\tfrac{x+n}{n}\right) \\
&= \lim_{n\to\infty} n(\ln(x+n) - \ln n) \\
&= \lim_{n\to\infty} n[\ln(y+n)]_{y=0}^{y=x} \\
&= \lim_{n\to\infty} n \int_0^x \tfrac{dy}{y+n} \\
&= \lim_{n\to\infty} \int_0^x \left(1 - \tfrac{y}{y+n}\right)\, dy \\
&= x - \int_0^x \lim_{n\to\infty} \tfrac{y}{y+n}\, dy \\
&= x - \int_0^x 0\ dy \\
&= x
\end{aligned}
$$

So $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$.

But also,

$$
\begin{aligned}
\lim_{n\to\infty} \left(1+\tfrac{x}{n}\right)^n &= \lim_{n\to\infty} \textstyle\sum_{r=0}^n \binom{n}{r} 1^{n-r} \left(\tfrac{x}{n}\right)^r \\
&= \lim_{n\to\infty} \textstyle\sum_{r=0}^n \tfrac{x^r n!}{n^r r!(n-r)!} \\
&= \lim_{n\to\infty} \left(1 + \textstyle\sum_{r=1}^n \tfrac{\prod_{i=1}^r (n-i+1)}{r!} \cdot \tfrac{x^r}{n^r}\right) \\
&= \lim_{n\to\infty} \left(1 + \textstyle\sum_{r=1}^n \tfrac{x^r}{r!} \cdot \prod_{i=1}^r \tfrac{n-i+1}{n}\right) \\
&= \lim_{n\to\infty} \left(1 + \textstyle\sum_{r=1}^n \tfrac{x^r}{r!} \cdot \prod_{i=1}^r \left(1 - \tfrac{i-1}{n}\right)\right) \\
&= 1 + \textstyle\sum_{r=1}^\infty \tfrac{x^r}{r!} \cdot \prod_{i=1}^r \left(1 - \lim_{n\to\infty} \tfrac{i-1}{n}\right) \\
&= \textstyle\sum_{r=0}^\infty \tfrac{x^r}{r!} \cdot 1 \\
&= \textstyle\sum_{n=0}^\infty \tfrac{x^n}{n!}
\end{aligned}
$$

Hence $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = \sum_{n=0}^\infty \frac{x^n}{n!} = e^x$ □

Proposition. $e \notin \mathbb{Q}$.

Proof. $1+\sum_{r=0}^{\infty}\frac{1}{\prod_{i=0}^{r}(n+2+i)}<\sum_{r=0}^{\infty}\frac{1}{2^r}=\frac{1}{1-\frac{1}{2}}=2.$

Hence $\frac{1}{(n+1)!}\left(1+\sum_{r=0}^{\infty}\frac{1}{\prod_{i=0}^{r}(n+2+i)}\right)=\sum_{r=1}^{\infty}\frac{1}{(n+r)!}<\frac{2}{(n+1)!}$

$\therefore$ from the **Lemma**, with $x=1$ for any $n\in\mathbb{Z}^+$, $e=\sum_{r=0}^{\infty}\frac{1}{r!}=\sum_{r=0}^{n}\frac{1}{r!}+R_n$ where

$R_n:=\sum_{r=1}^{\infty}\frac{1}{(n+r)!}$.

Now we have that $0<n!R_n<n!\cdot\frac{2}{(n+1)!}=\frac{2n!}{n!(n+1)}=\frac{2}{n+1}$.

If $n>1$, $\frac{2}{n+1}<1$ and so $0<n!R_n<\frac{2}{n+1}<1$.

Therefore $n!R_n\notin\mathbb{Z}^+$ .... $(*)$.

Notice that $n!,\frac{n!}{2!},\ldots,\frac{n!}{n!}\in\mathbb{Z}^+\ \therefore\ \sum_{r=0}^{n}\frac{n!}{r!}=n!\sum_{r=0}^{n}\frac{1}{r!}\in\mathbb{Z}^+$.

Now assume $e\in\mathbb{Q}$, i.e., that e is rational.

$e\in\mathbb{Q}\Rightarrow$ there exists $a,b\in\mathbb{Z}^+$ such that $e=\frac{a}{b}$ since $e>0$

$\Rightarrow$ if $n>b$ such that $b|n!$ so that $\frac{n!}{b}\in\mathbb{Z}^+$ then we have that

$\frac{n!a}{b}=n!e=n!\big(\sum_{r=0}^{n}\frac{1}{r!}+R_n\big)=n!\sum_{r=0}^{n}\frac{1}{r!}+n!R_n\in\mathbb{Z}^+$ since $a\in\mathbb{Z}^+$

$\Rightarrow n!R_n\in\mathbb{Z}^+$ since $n!\sum_{r=0}^{n}\frac{1}{r!}\in\mathbb{Z}^+$. But this contradicts $(*)$.

Hence the assumption that $e\in\mathbb{Q}$ is false. Hence $e\notin\mathbb{Q}$ $\square$

Method 3

From the **Lemma** of **Method 2**, $e^x=\sum_{i=0}^{\infty}\frac{x^i}{i!}=\left(\sum_{i=0}^{n}\frac{x^i}{i!}\right)+\frac{x^{n+1}e^{\xi}}{(n+1)!}$ for some $n\in\mathbb{Z}^+$ where $0<\xi<|x|$ by Taylor's Theorem†.

Now letting $x=1$ and assuming that e is a rational number $\frac{p}{q}$ in lowest terms where p and q are positive integers, we have that

$e=\frac{p}{q}=\left(\sum_{i=0}^{n}\frac{1}{i!}\right)+\frac{e^{\xi}}{(n+1)!}$ where $0<\xi<1$.

Now letting $n>q$, we have

---

†Taylor's Theorem states that if $f^{(n)}(x)$ is continuous in $[a,b]$ and differentiable in (a,b) and if x and x_0 are in (a,b) then there exists a point ξ strictly between x and x_0 such that $f(x)=\left(\sum_{i=0}^{n}\frac{f^{(i)}(x_0)(x-x_0)^i}{i!}\right)+\frac{f^{(n+1)}(\xi)(x-x_0)^{n+1}}{(n+1)!}$ where $f^{(i)}(x):=\frac{d^if(x)}{dx^i}$.

$n!e = n!\frac{p}{q} = \left(\sum_{i=0}^{n} \frac{n!}{i!}\right) + \frac{e^{\xi}}{n+1}$ ....(‡)

Now $\frac{e^{\xi}}{n+1}$ is a number strictly between 0 and 1, while every other term in (‡) is a positive integer. Thus assuming that e is rational, we arrive at the contradiction that an integer $\left(n!\frac{p}{q}\right)$ is equal to a non-integer. Hence reductio ad absurdum, e must be irrational. $\square$

Method 4

e is trancendental, therefore e is irrational $\square$

Reference.

NSW Board of Studies (2001),

http://www.boardofstudies.nsw.edu.au/hsc_exams/mathemat_ext2_01.pdf

Proof that π is irrational

Let $f_n(x) := \frac{x^n(1-x)^n}{n!}$ so that $0 < f_n(x) < \frac{1}{n!}$ for $0 < x < 1$ and

$f_n(x) = \frac{1}{n!}\sum_{i=n}^{2n} c_i x^i$ where $c_i \in \mathbb{Z}$.

Denote $\frac{d^k f_n(x)}{dx^k}$ by $f_n^{(k)}(x)$.

Then $f_n^{(k)}(0) = 0$ if $k < n$ or $k > 2n$ and

$f_n^{(n+i)}(x) = \frac{1}{n!}((n+i)!c_{n+i} + P_i(x))$ for some polynomial $P_i(x)$ of x for $i = 0, \ldots, n$.

So $f_n^{(n)}(0) = c_n \in \mathbb{Z}$, & $f_n^{(n+i)}(0) = \left(\prod_{j=1}^{i}(n+j)\right)c_{n+i} \in \mathbb{Z}$ for $i = 1, \ldots, n$.

So $f_n^{(k)}(0) \in \mathbb{Z}$ for all $k \in \mathbb{Z}^+$.

By symmetry, also therefore, $f_n^{(k)}(1) \in \mathbb{Z}$ for all $k \in \mathbb{Z}^+$.

Now if for some $a \in \mathbb{R}^+$, $n \geq 2a > 0$ then $\frac{a^{n+1}}{(n+1)!} = \frac{a}{n+1}\cdot\frac{a^n}{n!} \leq \frac{n}{2n+2} < \frac{n+1}{2n+2} = \frac{1}{2}\cdot\frac{a^n}{n!}$.

Similarly if $m \in \mathbb{Z}^+$ with $m \geq 2a$, then inductively, $\frac{a^{m+k}}{(m+k)!} < \frac{1}{2^k}\cdot\frac{a^m}{m!}$ for all $k \in \mathbb{Z}^+$.

For all $\xi > 0$ there exists k large enough such that $\frac{a^m}{m!\xi} < 2^k$.

$\therefore\ \frac{a^{m+k}}{(m+k)!} < \frac{1}{2^k}\cdot\frac{a^m}{m!} < \frac{1}{2^k}\cdot\xi\cdot 2^k = \xi$.

So for large enough n, $\frac{a^n}{n!} < \xi$ .... $(*)$.

Let $\xi = \frac{1}{\pi}$.

Now assume that $\pi = \frac{1}{\xi} \in \mathbb{Q}$, i.e., that π is rational. Then there exist $a, b \in \mathbb{Z}^+$ where a and b are perfect squares such that $\pi^2 = \frac{a}{b} \in \mathbb{Q}$.

Now define $g(x) := b^n \sum_{k=0}^{n}(-1)^k \pi^{2n-2k} f_n^{(2k)}(x)$.

Since $b^n\pi^{2n-2k} = b^n(\pi^2)^{n-k} = b^n\left(\frac{a}{b}\right)^{n-k} = a^{n-k}b^k \in \mathbb{Z}^+$
and since $f_n^{(2k)}(0), f_n^{(2k)}(1) \in \mathbb{Z}$, then $g(0), g(1) \in \mathbb{Z}$.

$g''(x) = b^n \sum_{k=0}^{n}(-1)^k \pi^{2n-2k} f_n^{(2k+2)}(x)$ and $(-1)^n f_n^{(2n+2)}(x) = 0$.

So $g''(x) + \pi^2 g(x) = b^n \pi^{2n+2} f_n(x) = \pi^2 a^n f_n(x)$.

Now define $h(x) := g'(x)\sin\pi x - \pi g(x)\cos\pi x$.

$$
\begin{aligned}
\text{Hence } h'(x) &= \pi g'(x)\cos\pi x + g''(x)\sin\pi x - \pi g'(x)\cos\pi x + \pi^2 g(x)\sin\pi x \\
&= (g''(x) + \pi^2 g(x))\sin\pi x \\
&= \pi^2 a^2 f_n(x)\sin\pi x.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\pi^2 \int_0^1 a^n f_n(x)\sin\pi x\ dx &= h(1) - h(0) \\
&= g'(1)\sin\pi - \pi g(1)\cos\pi - g'(0)\sin 0 + \pi g(0)\cos 0 \\
&= \pi(g(1) + g(0)).
\end{aligned}
$$

Thus $\pi \int_0^1 a^n f_n(x)\sin\pi x\ dx \in \mathbb{Z}$.

But $0 < f_n(x) < \frac{1}{n!}$ for $0 < x < 1$. So $0 < \pi a^n f_n(x)\sin\pi x < \frac{\pi a^n}{n!}$ for $0 < x < 1$.

Hence $0 < \pi \int_0^1 a^n f_n(x)\sin\pi x\ dx < \frac{\pi a^n}{n!}$ independent of n.

So for large enough n, where $\pi \int_0^1 a^n f_n(x)\sin\pi x\ dx \in \mathbb{Z}^+$, we can have from $(*)$ that

$0 < \pi \int_0^1 a^n f_n(x)\sin\pi x\ dx < \frac{\pi a^n}{n!} < \pi\xi = 1$.

But alas there is no positive integer between 0 and 1.

So $\pi^2 \notin \mathbb{Q}$, a and b cannot both be perfect square positive integers and $\pi \notin \mathbb{Q}$ $\square$